

NUMERICAL TREATMENT OF A
BRITTLE CRACK EXTENDING
AT NON - UNIFORM VELOCITY



HANS BERGKVIST

LUND INSTITUTE OF TECHNOLOGY
DIVISION OF SOLID MECHANICS
BOX 725

S-220 07 LUND, SWEDEN

1974



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41545309_0026

NUMERICAL TREATMENT OF A BRITTLE CRACK

EXTENDING AT NON-UNIFORM VELOCITY

by

HANS BERGKVIST

Division of Solid Mechanics

Lund Institute of Technology, Lund, Sweden

Report April 1974



SUMMARY

The problem concerning a central crack of finite length, loaded in the extensional mode and extending at non-uniform velocity has been treated numerically in an analysis of the corresponding mixed boundary value problem. An earlier work by BROBERG (1959) giving the displacement due to an impulsive line load on the surface of a semi-infinite, elastic, isotropic solid, has been extended to give a relation, in a double integral formulation, between the displacement and the stress distribution on the surface of the solid. The requirement of vanishing normal displacement on the parts of the surface that correspond to the plane of symmetry outside of the crack has given a series of Fredholm integral equations of the first kind for the stress distribution for a number of discrete times. The matrix formulation of these equations has been solved in terms of singular value decomposition and a weighted least squares solution. The stress distribution determined, the crack shape, the stress intensity factor and the energy release have been calculated. Material data for polymethylmethacrylate, PMM, have been chosen. Comparison is made between the results obtained and those of earlier experimental investigations and a quasi-dynamic analysis.

1. INTRODUCTION

The fundamental classical solution of the elastic, dynamic mode I problem is due to BROBERG (1960). In this work a semi-infinite solid $z \geq 0$ is subjected to a uniform pressure q_0 on the strip $|x| \leq v_0 t$ and to such surface loads on $|x| > v_0 t$ that the requirement of vanishing normal displacement on this part of the surface is met. This mixed boundary value problem is equivalent to that of a central crack in an infinite solid, expanding from zero initial length at constant velocity. The treatment of this constant velocity case forms the basis for a subsequent work by BROBERG (1964), giving the energy release. In BROBERG (1967) it is demonstrated that a crack of finite length initially being at rest, quickly will attain a high velocity of propagation. The acceleration phase is not investigated in detail.

The interest in the complete solution of the mode I problem, that is including an analysis of the acceleration phase, has been reflected in a number of papers of primarily theoretical nature, FREUND (1972), TSAI (1973) in later years. The solutions given in these papers, however, seem to be restricted in certain respects. So the solution in the first one is only valid as long as the crack-tip studied is unaffected of the stress-waves emanating from the other crack-tip. In practice this restriction is a severe one when cracks of finite length are concerned.

In TSAI's (1973) paper the solution, although claimed to be valid for an arbitrarily moving crack-tip, seems to be extremely difficult to evaluate when dealing with a crack moving in a more complex fashion than at a constant velocity or with a constant acceleration.

In an earlier work by the writer, BERGKVIST (1973) a simplified and what could be called a quasi-dynamic analysis has been carried out for the acceleration phase of a brittle crack of finite length in the extensional mode. The analysis has been applied numerically on crack motion in polymethylmethacrylate, PMM, and agrees well with experimental investigations by DULANEY and BRACE (1960).

2. THE FUNDAMENTAL EQUATIONS

Stress waves on the surface of a semi-infinite elastic solid, $z \geq 0$, subjected to an impulsive line load, have been studied by BROBERG (1959) and this analysis forms a basis for the formulation of the following equations. First consider the stress state on the surface $z = 0$

$$\sigma_z(x, t) = \frac{dI}{dy} \delta(x) \delta(t) \quad (1)$$

$$\tau_{zx} = \tau_{zy} = 0$$

where $\frac{dI}{dy}$ is the impulse per unit of length. Then the displacement $u_z(x, t)$ corresponding to this stress distribution becomes

$$u_z(x, t) = \frac{\frac{dI}{dy}}{\pi \rho c_d |x|} \cdot s \quad (2)$$

where

ρ is the density of the solid, c_d is the velocity of irrotational waves, and s is the non-dimensional displacement. When the irrotational and equivoluminal displacements are added the following expression is obtained

$$u_z(x, t) = - \frac{\frac{dI}{dy}}{4k^4 \pi \rho c_d |x|} \operatorname{Re} \left\{ \frac{\sqrt{\frac{c_d^2 t^2}{x^2} - 1}}{\left(\frac{c_d^2 t^2}{x^2} - \frac{1}{2k^2} \right)^2 - \frac{c_d^2 t^2}{x^2} \sqrt{\left(\frac{c_d^2 t^2}{x^2} - 1 \right) \left(\frac{c_d^2 t^2}{x^2} - \frac{1}{k^2} \right)}} \right\} U(t) \quad (3)$$

where

$k = \frac{c_r}{c_d}$ is the ratio between the velocities of equivoluminal and irrotational

waves

$\text{Re} \{ \}$ denotes the real part of the expression within the brackets

$$U(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad \text{i.e. the unit-step-function}$$

Thus in compact form

$$\sigma_z(x,t) = \frac{dI}{dy} \delta(x) \delta(t) \quad (4)$$

gives

$$u_z(x,t) = -c \frac{\frac{dI}{dy}}{|x|} \text{Re} \left\{ f \left(\frac{c_d t}{x} \right) \right\} U(t) \quad (5)$$

where

$$c = \frac{1}{4k^4 \pi \rho c_d}$$

Then

$$\sigma_z(x,t) = \frac{dI}{dy} \delta(x-x_0) \delta(t-t_0) \quad (6)$$

gives

$$u_z(x,t) = -c \frac{\frac{dI}{dy}}{|x-x_0|} \text{Re} \left\{ f \left(\frac{c_d (t-t_0)}{x-x_0} \right) \right\} U(t-t_0) \quad (7)$$

Analogously

$$\sigma_z(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_z(x_0, t_0) dx_0 dt_0 \delta(x-x_0) \delta(t-t_0) \quad (8)$$

gives

$$u_z(x, t) = -c \int_{-\infty}^t \int_{-\infty}^{\infty} \frac{\sigma_z(x_0, t_0)}{|x-x_0|} \operatorname{Re} \left\{ f \left(\frac{c_d(t-t_0)}{x-x_0} \right) \right\} dx_0 dt_0 \quad (9)$$

Rewriting this last equation in such a way that x_0 runs from 0 to ∞ and using the notation

$$\operatorname{Re} \left\{ f \left(\frac{c_d(t-t_0)}{x-x_0} \right) \right\} = g \left(\frac{c_d(t-t_0)}{x-x_0} \right) = h \left(\frac{x-x_0}{c_d(t-t_0)} \right) \quad (10)$$

one obtains for $x \geq 0$

$$u_z(x, t) = -c \int_{-\infty}^t \int_0^{\infty} \sigma_z(x_0, t_0) \left[\frac{g \left(\frac{c_d(t-t_0)}{x-x_0} \right)}{|x-x_0|} + \frac{g \left(\frac{c_d(t-t_0)}{x+x_0} \right)}{x+x_0} \right] dx_0 dt_0. \quad (11)$$

Having introduced some reference length a , it is advantageous to use non-dimensional quantities

$$\xi = \frac{x}{a}; \quad \xi_0 = \frac{x_0}{a}$$

$$\tau = \frac{c_d t}{a}; \quad \tau_0 = \frac{c_d t_0}{a}$$

and

$$v = \frac{u}{a}$$

Thus (11) can be written

$$v_z(\xi, \tau) = -c_1 \int_{-\infty}^{\tau} \int_0^{\infty} \sigma_z(\xi_0, \tau_0) G(\tau - \tau_0, \xi, \xi_0) d\xi_0 d\tau_0 \quad (12)$$

where

$$c_1 = \frac{1}{4k^4 \pi \rho c_d^2} \quad (13)$$

and

$$G(\tau - \tau_0, \xi, \xi_0) = \frac{g\left(\frac{\tau - \tau_0}{\xi - \xi_0}\right)}{|\xi - \xi_0|} + \frac{g\left(\frac{\tau - \tau_0}{\xi + \xi_0}\right)}{(\xi + \xi_0)} \quad (14)$$

This expression (12) gives a general relation between the stress and the displacement distribution on the surface of a elastic, isotropic semi-infinite solid.

Supposing that the stresses start acting at $t = 0$, (12), dropping index z , reads

$$v(\xi, \tau) = -c_1 \int_0^{\tau} \int_0^{\infty} \sigma(\xi_0, \tau_0) G(\tau - \tau_0, \xi, \xi_0) d\xi_0 d\tau_0 \quad (15)$$

where the meaning of c_1 and G is unaltered from (13) and (14) respectively.

As (15) can be written

$$v(\xi, \tau) = -c_1 \int_0^{\tau} f(\tau_0) d\tau_0$$

Some quadrature formula can be applied on the integral, giving

$$v(\xi, \tau) \approx -c_1 \sum f(\bar{\tau}_i) \Delta \tau_i$$

where

$$\bar{\tau}_i = 0.5(\tau_i + \tau_{i-1})$$

But

$$f(\bar{\tau}_i) = \int_0^{\infty} \sigma(\xi_0, \bar{\tau}_i) G(\tau - \bar{\tau}_i, \xi, \xi_0) d\xi_0$$

Thus the displacement distribution at the time τ_n is given from

$$v(\xi, \tau_n) \approx -c_1 \sum_{i=1}^n \int_0^{\infty} \sigma(\xi_0, \bar{\tau}_i) G(\tau_n - \bar{\tau}_i, \xi, \xi_0) d\xi_0 \Delta\tau_i \quad (16)$$

The subsequent analysis in this paper rests on the use of a basic element which will be called a stress box or simply a box. A box is a uniform pressure q acting on the portion from 0 to χ_i during the time interval $\Delta\tau_i$ at time $\bar{\tau}_i$. This notation has among other aspects also the conceptual convenience that if the χ_i 's increase at a constant fraction β of τ_i , it becomes natural to look upon the box as expanding at a constant relative velocity β .

Thus the displacement v_j due to box j becomes

$$v_j(\xi, \tau_n) \approx -c_1 q_j \sum_{i=1}^n \int_0^{\chi_i} G(\tau_n - \bar{\tau}_i, \xi, \xi_0) d\xi_0 \Delta\tau_i \quad (17)$$

As integrals of the type

$$\int_{\xi_a}^{\xi_b} G(\tau_n - \bar{\tau}_i, \xi, \xi_0) d\xi_0 \quad (18)$$

are comparatively easily evaluated, cf Appendix I, the calculation of the integrals making up (17) is completely straightforward.

3. THE DISPLACEMENT DISTRIBUTION PROBLEM

In this section the displacement distribution due to a box of initial length equal to zero and expanding at a constant fraction β of the velocity of irrotational waves is calculated from (17).

Thus

$$x_i = \beta \cdot \frac{2i-1}{2i} \cdot \tau_i \quad (19)$$

It is clearly seen from Figs. 1. and 2. that the resulting displacement distributions become more smooth as the number of time steps n increase. The steady state seems to be well approximated when $n=100$ as an increase to $n=200$ does not significantly change the curves. The calculations have been performed for a material having Poisson's ratio $\nu=0.25$, i.e. $k^2=1/3$. This choice of parameter is made since it permits a direct comparison between the results found and the exact displacement distribution u_z that can be derived from an analytical expression for the second derivative with respect to time of the displacement, $\partial^2 u_z / \partial \tau^2$. The expression for this acceleration is given by BROBERG (1960). The calculation of u_z from $\partial^2 u_z / \partial \tau^2$ has been carried out in Appendix II.

The resulting curves from (17) for $n=200$, given in Fig. 2, are impossible to distinguish from the exact distribution from Appendix II on the scale of the figure.

4. THE STRESS DISTRIBUTION PROBLEM

While the solution of the displacement distribution problem in the previous section in a way is a matter of handling a series of integrals, the problem in this section is highly more complex. Here the stress distribution is sought on the plane of symmetry along a central crack moving in a plane of an infinite solid loaded in the extensional mode. The analysis is performed in terms of a study of the equivalent mixed boundary value problem of a semi-infinite body, compare Fig. 3. In this case the stress has a prescribed magnitude on the part of the surface that corresponds to the interior of the crack, whereas the displacement is prescribed to zero on the other parts of the surface which consequently correspond to the plane of symmetry along the crack.

Looking at (12) or (15) it is clear that either of these equations give means of determining the unknown stress distribution on the parts of the surface where the displacement is equal to zero. The formulation could be called a double integral equation of the first kind. Using a quadrature formula with respect to time, (12) or (15) are replaced by (16) which can be looked upon as an ordinary Fredholm integral equation of the first kind. The solution of such equations is, generally speaking, an extremely difficult thing from an analytical as well as a computational point of view. In Appendix III some aspects on these matters are treated.

Before going to the analysis of the general problem, that is of a crack moving in an arbitrary but yet prescribed fashion, the stress distribution on the plane of symmetry along a crack of zero initial length expanding at constant velocity is sought.

The Constant Velocity Case

It was found in the previous section that the displacement distributions $v_j(\xi, \tau_n)$ from (17) due to a box of zero initial length and expanding at a constant fraction β_j of the velocity of irrotational waves, will approach a steady state distribution after a number of time increments, cf. Figs. 1. and 2. Furthermore it was demonstrated that the steady state distributions agree very well with the corresponding displacement distributions of the constant velocity case treated in Appendix II.

Thus

$$\frac{v_j(\xi, \tau_n)}{\tau_n} \longrightarrow v_j(\xi) \quad (20)$$

and, as v_j is dependent on β_j

$$v_j = q_j \bar{v}(\xi, \beta_j) \quad (21)$$

Knowledge of the functions $\bar{v}$ for different β_j and ξ should be sufficient to enable the calculation of the stress distribution in the plane of a crack in the extensional mode, extending at a constant velocity.

The situation is pictured in Fig. 4. Here a number of boxes of amplitude q_j and expansion velocity β_j act over the portions

$$|\xi| \leq \beta_j$$

where

$$\beta_0 < \beta_j \leq 1$$

and where β_0 denotes the relative velocity of the expanding crack.

Thus the resulting displacement v at a point ξ_i can be written

$$v(\xi_i) = \sum_{j=1}^n q_j \bar{v}(\xi_i, \beta_j) - q_0 \bar{v}(\xi_i, \beta_0) \quad (22)$$

or, introducing $v_{ij} = \bar{v}(\xi_i, \beta_j)$

$$v(\xi_i) = \sum_1^n v_{ij} q_j - q_0 v_{i0} \quad (23)$$

To be able to calculate the n unknown amplitudes q_j , it is necessary to introduce the appropriate boundary conditions.

Thus

$$v(\xi_i) = 0 \quad \text{for} \quad \beta_0 \leq \xi_i \leq 1 \quad (24)$$

and furthermore

$$q_0 = q_{00} + \sum_1^n q_j \quad (25)$$

where q_{00} is the pressure acting on the parts of the surface that correspond to the interior of the crack.

Introducing (24) and (25) in (23), the latter equation reads

$$\sum_1^n v_{ij} q_j = (q_{00} + \sum_1^n q_j) v_{i0} \quad (26)$$

or

$$\sum_1^n (v_{ij} - v_{i0}) q_j = v_{i0} q_{00} \quad (27)$$

The stress distribution is given by Q_j from

$$Q_m = \sum_{j=m}^n q_j \quad (28)$$

cf. Fig. 5., and thus (26) could be rewritten in terms of Q_j directly.

It follows from (28) that

$$q_m = Q_m - Q_{m+1} \quad m=1,2,\dots, n-1 \quad (29)$$

$$q_n = Q_n$$

thus (26) reads

$$\sum_{j=1}^{n-1} v_{ij} (Q_j - Q_{j+1}) + v_{in} Q_n = (q_{oo} + Q_1) v_{io} \quad (30)$$

or

$$\sum_{j=1}^n (v_{ij} - v_{ij-1}) Q_j = q_{oo} v_{io} \quad (31)$$

The system of equations (27) has been solved in accordance with the outlines in Appendix III, i.e. through a weighted least squares solution, performed with singular value decomposition. The number of unknowns n has depended on the value of β_o in such a way that $n=(1-\beta_o) \cdot 20$ for $\beta_o=0.05(0.05)0.50$. The number of equations has been $(1-\beta_o) \cdot 100$.

The equations have been weighted in two ways. In the first solution each equation was given the same weight and in the second solution the weight

of each equation was determined by the reciprocal of the amplitude of the right hand side of (27). Referring to Appendix III this latter choice of weights can be interpreted as the standard deviation of each component of the right hand side is assumed to be proportional to the absolute value of the component itself and that the equations are weighted by the reciprocal of these standard deviations.

This method seems to give results that are superior to the ones obtained from the first method. The problem has shown a tendency to become increasingly ill-conditioned when smaller β_0 -values are treated, which is reflected in a dramatic increase of the ratio between the greatest and the smallest singular values.

Some typical results are given in Fig. 6. BROBERG (1960) has given an expression for the stress distribution for this problem. The expression is evaluated numerically as described in Appendix IV. As seen from Fig. 6. the agreement between the two methods of calculation is good.

Non-uniform velocity case

When the more general case i.e. the situation when the crack is expanding in an arbitrary but yet prescribed fashion is to be treated, the self-similarity of the constant velocity case is no longer present. Thus all quantities must be thought of as functions of time and space, and the discretization with respect to time has to be made in addition to the spatial discretization. In the present analysis a number of fundamental time increments have been lumped together to make up a set of "check times" of Fig. 7, at which the boundary condition of vanishing normal displacement on the parts of the surface that correspond to the plane of symmetry outside the crack is satisfied.

By generalizing the equations of the previous subsection, the system of equations giving the stress distribution $Q_j^{(1)}$ at check-time $\tau(m(1))$ reads

$$\sum_{j=1}^n \left(\frac{m(1)}{1} v_{ij}^{m(1)} - \frac{m(1)}{1} v_{ij-1}^{m(1)} \right) Q_j^{(1)} = q_{oo} \frac{m(1)}{1} v_{io}^{m(1)} \quad (32)$$

where the notation $c_{b ij}^d v_{ij}^d$ stands for

$$c_{b ij}^d v_{ij}^d = \sum_{k=b}^d \int_0^{z_j^{(k)}} G(\tau_d - \bar{\tau}_k, \xi_i, \xi_o) d\xi_o \Delta\tau_k \quad (33)$$

Furthermore

$$\xi_i \geq z_o(m(1)) \quad (34)$$

where $z_o(m(1))$ is the position of the crack-tip at check time $\tau(m(1))$.

Solving (32) in the fashion described above and in Appendix III, the result can be given as

$$Q_j^{(1)} = c_j^{(1)} q_{oo} \quad (35)$$

Proceeding to check time $\tau(m(2))$, the right hand side of the equation corresponding to (32) does not only contain the effects on the displacement of the crack motion in the time segment from $\tau(m(1))$ to $\tau(m(2))$. In addition to this the stress distribution present from zero up to time $\tau(m(1))$, gives an additional deflection of the surface. The stress distribution $Q_j^{(2)}$ at time $\tau(m(2))$ is determined so that the total net deflection is equal to zero on the parts of the boundary that correspond to the plane of symmetry outside the crack.

Thus

$$\begin{aligned} \sum_{j=1}^n ({}^{m(2)}_{1+m(1)} v_{ij}^{m(2)} - {}^{m(2)}_{1+m(1)} v_{ij-1}^{m(2)}) Q_j^{(2)} &= q_{oo} {}^{m(2)}_{1+m(1)} v_{io}^{m(2)} + \\ &+ q_{oo} \left\{ {}^{m(1)}_1 v_{io}^{m(2)} - \sum_{j=1}^n ({}^{m(1)}_1 v_{ij}^{m(2)} - {}^{m(1)}_1 v_{ij-1}^{m(2)}) c_j^{(1)} \right\} \end{aligned} \quad (36)$$

where

$$\xi_i \geq z_o(m(2))$$

Using an even more compact notation, (36) reads

$${}^2_V {}^2_C {}^2_C = {}^2_B {}^2_B + {}^2_B {}^2_B - {}^2_V {}^2_C {}^2_C \quad (37)$$

where the meaning of the different terms is obvious.

Thus at check time $\tau(m(n))$ the equation reads

$$n_V C_n = n_{B_n} - \left(\sum_{i=1}^{n-1} n_{V_i} C_i - n_{B_i} \right) \quad (38)$$

with the boundary condition expressed as

$$\xi_i \geq z_0(m(n)) \quad (39)$$

The writer, BERGKVIST (1973), has performed a quasi-dynamic analysis of the acceleration phase of a brittle crack of finite length. The analysis been applied numerically on crack motion in polymethylmethacrylate, PMM. The results include, in non-dimensional form, three functional relations between length, velocity and time. The length vs. time relation, pictured in Fig.8, has been used in the non-uniform velocity case in the present work. A number of series of calculations have been carried out, each comprising 10 check-times, typically chosen as 20(20)200, 40(40)400 and 80(80)800. These non-dimensional quantities are expressed in terms of $c_d t/a_0$ where c_d is the velocity of irrotational waves and $2a_0$ is the original crack length.

Some examples of stress distributions calculated for this crack motion are found in Fig.9.

5. THE CRACK SHAPE

The problem treated in the present paper is, as was pointed out earlier, essentially a mixed boundary value problem. This means that the displacement is prescribed on certain parts of the boundary and unknown on the rest while the reverse conditions apply for the stress. Having chosen to evaluate the unknown stress distribution as was done in the previous section, the success in this respect also gives the solution for the unknown displacement, i.e. the shape of the crack.

A numerical evaluation of the crack shape for the constant velocity case is done comparatively easily. An example of such a computation is given in Fig.10. Here $\beta_o = 0.30$ and the crack contour is shown for two cases where a different number of the singular values are retained in the solution of the corresponding stress distribution. The ratio between the shorter and longer diagonal of the crack gives $1.42 q_{oo}/E$ to $1.45 q_{oo}/E$.

BROBERG (1960) has given the expression

$$\frac{Q(\beta_o)}{\rho c_d^2 \beta_o^2} \quad (40)$$

for this ratio, where

$$Q(\beta_o) = q_{oo} \frac{\beta_o^2 (1 - \beta_o^2)}{g(\beta_o)} \quad (41)$$

and the expression for $g(\beta_o)$ is found in Appendix IV.

Furthermore in this plane strain case

$$\rho c_d^2 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \quad (42)$$

where E is Young's modulus.

Evaluating (40), using (41) and (42) for

$\beta_0 = 0.3$, $\nu = 1/4$ will give $1.43 q_{00}/E$

The agreement between the two methods of calculation in this respect therefore can be considered as very good.

The numerical evaluation of the crack shape in the general case will involve a sizable although straightforward amount of computational work.

In fact

$$v(\xi_i, \tau(m(n))) = - \sum_{i=1}^n (n_{V_i} C_i - n_{B_i}) \quad (43)$$

will give the resulting displacement at a point ξ_i and check-time $\tau(m(n))$. Choosing $\xi_i \leq z_0(m(n))$ the values of v will give the shape of the crack, and looking at points $\xi_i > z_0(m(n))$ will tell how well the condition of vanishing displacement on this part of the boundary is realized.

The crack shape has been calculated for the crack motion in PMM, described in the previous section for the check-times $c_d t/a_0 = 40(40)400$. The ratio between the shorter and longer diagonals of the crack has been evaluated and plotted vs mean crack-tip velocity in the time segment, cf Fig.11. For comparison purposes this ratio in the constant velocity case, see above, has been calculated as well for PMM-data, i.e. $\nu = 0.4$. This curve is also plotted in Fig.11. As seen the accelerating crack is sharper than the one moving at constant velocity, just as would be expected.

6. THE STRESS SINGULARITY

To be able to calculate the stress intensity factor and the release of energy from the stress field due to an arbitrarily moving crack, some knowledge of the stress singularity at the crack-tip is vital. NILSSON (1972) has studied this problem in formulating the wave equations in a polar coordinate system having its origin at the arbitrarily moving crack-tip. His results, although not a conclusive proof, indicate that the angular distribution of the singularity should depend only on the instantaneous crack-tip velocity.

BROBERG (1974) has suggested the following outline for the handling of this problem, cf Fig.12. Consider some region S surrounding the tip of an arbitrarily moving crack. The state of stress within this region is governed by a set of differential equations and by the appropriate boundary conditions. The boundary conditions, however, are not completely known, but it may be assumed that the stress distribution on Γ_0 is symmetric and furthermore that the crack surfaces are traction free. Now suppose that during some increment of time Δt the crack-tip moves in a prescribed way from a to $a + \Delta a$, where Δa is a fraction of some typical dimension of the region considered.

Since the existence of a singularity at the crack-tip is a natural consequence of the requirement of a continuous and finite flow of energy towards the crack-tip, the picture drawn above looks essentially the same if a considerably smaller region S_1 around the crack-tip is investigated. In particular it is possible to choose the size of the region so small that the crack will travel Δa_1 during the time increment Δt_1 at a velocity the deviation of which is inferior to any arbitrarily specified quantity, if the necessary assumption is made that the crack moves in a continuous way.

As regards the stress distribution on Γ_{σ_1} it may be assumed that this takes about the same shape regardless of the exact shape of the symmetric stress distribution on Γ_{σ} only S_1 is small enough in analogy with the conditions in the static case.

It may thus be concluded that the angular distribution of the stress singularity is dependent only on the instantaneous crack-tip velocity and that it consequently corresponds to the known one in the steady state case, cf BROBERG (1964).

7. THE STRESS INTENSITY FACTOR

The stress intensity factor, that is the strength of the inverse square-root singularity in the stress distribution at the crack-tip is known in analytical form in the constant velocity case, BROBERG (1964).

Denoting the relative velocity by β_0 , the stress intensity factor $A(\beta_0)$ has the form

$$A(\beta_0) = \frac{q_0 \sqrt{(1-\beta_0^2)}}{\sqrt{2\beta_0} \beta_0 g(\beta_0)} \left[4k^3 \sqrt{(1-\beta_0^2)(k^2-\beta_0^2)} - (\beta_0^2 - 2k^2)^2 \right] \sqrt{c_d t} \quad (44)$$

where, as before, k is the ratio between the basic wave velocities, and g is found in Appendix IV.

Normalizing $A(\beta_0)$ by a division by the corresponding static stress intensity factor, i.e. by $q_0 \sqrt{\beta_0 c_d t/2}$, the result reads

$$\frac{A(\beta_0)}{A(0)} = \frac{\sqrt{1-\beta_0^2}}{\beta_0^2 g(\beta_0)} \left[4k^3 \sqrt{(1-\beta_0^2)(k^2-\beta_0^2)} - (\beta_0^2 - 2k^2)^2 \right] \quad (45)$$

Here the right hand side goes to 1 as β_0 becomes small as would be expected. Furthermore it goes to zero as β_0 approaches the Rayleigh wave velocity.

A plot of the right hand side of (45) is found in Fig.13. In this diagram the results of the calculations of the corresponding quantity in the constant velocity case are given as well. Material data have been $k^2=1/3$, i.e. $v=0.25$. As seen the agreement between these calculations and the exact curve is good.

Assuming that the same kind of singularity is present also in the non-uniform velocity case, cf section 6, the stress intensity factor can be calculated from the stress distribution at any check-time.

Thinking about the stress distribution in terms of stress boxes, as described in section 2, it has shown convenient to use a number of boxes of fixed length immediately adjacent to the crack-tip. The length of the first box in the time segment $\tau(m(i))$ to $\tau(m(i+1))$ has been chosen as $0.1 z_0(m(i+1))$, i.e. one tenth of the crack length at the end of the time segment.

Numerical results from the cases treated in the calculation of the stress distributions are given in Fig.14. The stress intensity factors obtained are normalized by a division by the corresponding static stress intensity factor for the original crack length.

Looking at the magnitude of the stress intensity factor at a given velocity, it is seen from Fig.14 that the highest value is obtained from the case having the lowest acceleration which is what would be expected.

8. THE ENERGY RELEASE

Referring to the two previous sections, the strength $A(\beta_o)$ of the inverse square-root singularity in the stress distribution at the crack-tip is given by (44).

The coefficient $B(\beta_o)$ for the square-root term in the displacement distribution is given by BROBERG (1964) as

$$B(\beta_o) = \frac{q_o k^2 (1 - \beta_o^2) \sqrt{2\beta_o}}{G g(\beta_o)} \sqrt{c_d t} \quad (46)$$

where G is the shear modulus.

In the same reference the release of energy per unit of time at one crack-tip is calculated. Thus

$$\frac{\partial W}{\partial t} = \frac{1}{2} \pi AB c_d \beta_o \quad (47)$$

But

$$a = c_d \beta_o t$$

and so

$$\frac{\partial W}{\partial a} = \frac{1}{2} \pi AB \quad (48)$$

Since the stress intensity factor $A(\beta_o)$ can be estimated from the calculations as described in the previous section, it is desirable to be able to express B in terms of A .

Relating (44) and (46), it is easily found that

$$\frac{B}{A} = \frac{k^2 \sqrt{1-\beta_o^2} \cdot 2\beta_o^2}{G \left[4k^3 \sqrt{(1-\beta_o^2)(k^2-\beta_o^2)} - (\beta_o^2 - 2k^2)^2 \right]} \quad (49)$$

which leads to

$$\frac{\partial W}{\partial a} = \frac{1}{2} \pi A^2 \cdot \frac{k^2 \sqrt{1-\beta_o^2} \cdot 2\beta_o^2}{G \left[4k^3 \sqrt{(1-\beta_o^2)(k^2-\beta_o^2)} - (\beta_o^2 - 2k^2)^2 \right]} \quad (50)$$

Suppose that the original crack length is a_o and that the load is q_o . Then at the onset of crack extension, i.e. for $\beta_o = 0$, (50) reads

$$\left(\frac{\partial W}{\partial a} \right)_o = \frac{1}{2} \pi \cdot q_o^2 \frac{a_o}{2} \cdot \frac{1}{G(1-k^2)} \quad (51)$$

Normalizing loads with respect to q_o and crack length with respect to a_o (50) and (51) combine to

$$\frac{\left(\frac{\partial W}{\partial a} \right)_D}{\left(\frac{\partial W}{\partial a} \right)_o} = A_D^2(\beta_o) \cdot 2 \cdot F(\beta_o) \quad (52)$$

where $A_D(\beta_o)$ is the value of the normalized stress intensity factor, and

$$F(\beta_o) = \frac{(1-k^2)k^2 \sqrt{1-\beta_o^2} \cdot 2\beta_o^2}{4k^3 \sqrt{(1-\beta_o^2)(k^2-\beta_o^2)} - (\beta_o^2 - 2k^2)^2} \quad (53)$$

The quantities in (52) have been evaluated for the three cases of crack motion plotted in Fig.8. The results are given in Fig.15. For comparison purposes the normalized fracture energy $\Gamma(\dot{a}/c_d)$ for PMM as determined by BERGKVIST (1973) from experiments on the long strip configuration by PAXSON and LUCAS (1972), is given as well.

As was mentioned before, the crack motion giving curve (1) was obtained by the writer BERGKVIST (1973) in a quasi-dynamic analysis using PAXSON'S and LUCAS'S data. The agreement between curve (1) and PAXSON'S and LUCAS'S seems to be comparatively good for lower velocities, while curve (2) turns out to agree better at higher velocities. As the velocity in the crack motion on which curve (2) is based is about ten percent higher, these investigations seem to indicate that the quasidynamic analysis underestimates the crack velocity somewhat, roughly on the ten percent level at higher velocities.

These results agree with the conclusions drawn from a comparison between the results from the writer's quasi-dynamic analysis and those of experiments by DULANEY and BRACE (1960), cf. BERGKVIST (1973).

9. DISCUSSION

Ever since the appearance of the complete solution of the dynamic mode I problem for the constant velocity case, BROBERG (1960), interest has been more or less intensely focused on the natural extension of this problem, i.e. to include the solution of the general dynamic crack problem.

The writer's present numerical treatment of this problem in terms of a series of Fredholm integral equations of the first kind has been successful, primarily due to the use of modern computational methods to obtain the solution in the least squares sense of the associated matrix equations. Furthermore the series of solutions has proved to be numerically stable.

There is a number of factors that are likely to have influence on the solution, such as the number of singular values retained, the method of weighting the equations, the number of unknowns and so on. Thus some scatter in the individual results may be expected. Taking the overall picture into consideration the effects of the scatter however become less important. To illustrate this, consider Fig. 11. This diagram shows that the accelerating crack is sharper than the one running at a constant velocity for all the velocities investigated. Looking at the whole series, it may be concluded that the shape of the accelerating crack is approaching that of the constant velocity crack for higher velocities.

Looking at the results in this way, Fig. 15. becomes highly interesting. This figure shows that there is a significant difference in the energy release for cracks the speed of which deviate as little as ± 10 percent. Using the equality between the release of energy and some velocity dependent material property γ_F expressing the dissipative processes in the neighbourhood of the advancing crack-tip as a fracture criterion, cf. BERGKVIST (1973), the

correct crack motion should be the one that gives $\left(\frac{\partial W}{\partial a}\right)_D \bigg/ \left(\frac{\partial W}{\partial a}\right)_O = \Gamma(\dot{a}/c_d)$

Due to the sensitivity in the energy release on the way the crack moves, it may be concluded that a crack motion giving an energy release relation reasonably close to $\Gamma(\dot{a}/c_d)$ should be very near the correct one.

Looking at the results in Fig. 15. it may therefore be concluded that a quasi-dynamic analysis is likely to give results that come close to the correct ones. Trying to quantify things, the quasi-dynamic analysis seems to underestimate the crack velocity on the ten percent level at higher velocities, which is in agreement with experimental investigations, cf.

BERGKVIST (1973).

Finally it should be pointed out that in the present numerical routines the crack length vs. time is a free input parameter. Thus any experimental results could be used here since also the ordinary material properties as Young's modulus, Poisson's ratio and the density could be chosen at will.

Acknowledgments

The writer would like to thank Professor K.B.Broberg for numerous invaluable discussions and encouraging support during the course of this work. Financial support from the Swedish Board for Technical Development is gratefully acknowledged.

REFERENCES

- ABRAMOWITZ, M. and
STEGUN, I. A. 1970 Handbook of Mathematical
Functions p.591
U.S. Dept. of Commerce,
Washington D.C.
- BAKUSHINSKII, A. B 1965 USSR Comp. Maths and Math Phys.
(translation from Zh.vychisl.
Mat. mat. Fiz.) 5, 4, 226.
- BERGKVIST, H 1973 J. Mech. Phys. Solids, 21, 229.
- BOULLION, T. L. and
ODELL, P. L. 1971 Generalized Inverse Matrices
p.42 Wiley, New York.
- BROBERG, K. B 1959 Trans. Royal Institute of
Technology, 139, Stockholm
- 1960 Arkiv för Fysik, 18, 10, 159.
- 1964 J. Appl. Mech, 31, 546.
- 1967 Recent Progress in Applied
(edited by BROBERG, B., HULT, J.
and NIORDSON, F.) p.125
Almqvist & Wiksell, Stockholm
- 1974 Private communication
- DULANEY, E. N. and
BRACE, W. F. 1960 J. Appl. Phys. 31, 2233.
- FOX, L. 1962 Numerical Solutions of Ordinary
and Partial Differential Equa-
tions p.159.
Pergamon Press, New York.
- FREUND, L. B. 1972 J. Mech. Phys. Solids 20, 129
- GOLUB, G. H. and
REINSCH, C. 1969 Technical report CS 133,
Computer Science Dept., Stanford
University.

- | | | |
|---------------------------------|------|---|
| HAMSTAD, M.A. and
BOGY, D.B. | 1972 | Int.J.Solids Structures, <u>8</u> , 1. |
| HANSON, R.J. | 1971 | SIAM J.Numer.Anal. <u>8</u> , 616. |
| NILSSON, F. | 1972 | On the stress singularity
at an arbitrarily moving
crack tip. Royal Institute
of Technology, Stockholm |
| PAXSON, T.L. and
LUCAS, R.A. | 1972 | An Experimental Investigation
of the Velocity Characteristics
of a Fixed Boundary Model.
Lehigh University, Bethlehem,
Pennsylvania |
| SÖDERSTRÖM, T. | 1970 | Computer Routines Library.
Division of Automatic Control.
Lund Institute of Technology
Lund. |
| TRICOMI, F.G. | 1970 | Integral Equations, p.143.
Interscience, New York. |
| TSAI, Y.M. | 1973 | Int.J.Solids Structures, <u>9</u> , 625. |

Appendix INumerical evaluation of $\int G(\tau_n - \bar{\tau}_i, \xi, \xi_0) d\xi_0$

Referring to (14) and (10), it is immediately clear that

$$\int_{\xi_a}^{\xi_b} G(\tau_n - \bar{\tau}_i, \xi, \xi_0) d\xi_0 = \int_{\xi_a}^{\xi_b} \left[\frac{h\left(\frac{\xi}{\Delta\tau_{ni}} - \frac{\xi_0}{\Delta\tau_{ni}}\right)}{\left|\frac{\xi}{\Delta\tau_{ni}} - \frac{\xi_0}{\Delta\tau_{ni}}\right| \Delta\tau_{ni}} + \frac{h\left(\frac{\xi}{\Delta\tau_{ni}} + \frac{\xi_0}{\Delta\tau_{ni}}\right)}{\left(\frac{\xi}{\Delta\tau_{ni}} + \frac{\xi_0}{\Delta\tau_{ni}}\right) \Delta\tau_{ni}} \right] d\xi_0 \quad (\text{AI:1})$$

where $\Delta\tau_{ni} = \tau_n - \bar{\tau}_i$

Introducing the quantities

$$\Psi = \frac{\xi}{\Delta\tau_{ni}} ; \quad \Psi_0 = \frac{\xi_0}{\Delta\tau_{ni}} ; \quad \Psi_a = \frac{\xi_a}{\Delta\tau_{ni}} ; \quad \Psi_b = \frac{\xi_b}{\Delta\tau_{ni}}$$

(AI:1) reads

$$\int_{\Psi_a}^{\Psi_b} \frac{h(\Psi - \Psi_0)}{|\Psi - \Psi_0|} + \frac{h(\Psi + \Psi_0)}{(\Psi + \Psi_0)} d\Psi_0 \quad (\text{AI:2})$$

or, since h is symmetric

$$\int_{\Psi_a - \Psi}^{\Psi_b - \Psi} \frac{h(\Phi)}{|\Phi|} d\Phi + \int_{\Psi_a + \Psi}^{\Psi_b + \Psi} \frac{h(\Phi)}{\Phi} d\Phi \quad (\text{AI:3})$$

Thus the basic problem is the evaluation of an integral

$$I = \int_{X_1}^{X_2} \frac{h(X)}{X} dX \quad (\text{AI:4})$$

where $X_2 > X_1 \geq 0$

It is pointed out in Appendix II that the exact evaluation of integrals like (AI:4) is simplified if the restriction is imposed that the denominator in h has simple real zeroes. In this appendix, however, this restriction has to be removed since a numerical integration of G is desired, regardless of the actual value of Poisson's ratio ν or, which is the same thing, the value of the ratio k between the basic wave velocities.

Using (3) (AI:4) can be written

$$I = \int_{X_1}^{X_2} \frac{h(X)}{X} dX = \int_{X_1}^{X_2} \frac{\sqrt{1/X^2 - 1} (1/X^2 - 1/2k^2)^2}{N(1/X^2) \cdot X} U(|1/X| - 1) dX +$$

$$+ \int_{X_1}^{X_2} \frac{1/X^2 (1/X^2 - 1) \sqrt{1/X^2 - 1/k^2}}{N(1/X^2) X} U(|1/X| - 1/k) dX \quad (\text{AI:5})$$

where

$$N(1/X^2) = (1/X^2 - 1/2k^2)^4 - 1/X^4 (1/X^2 - 1)(1/X^2 - 1/k^2) =$$

$$= (1 - 1/k^2)(1/X^2 - A)(1/X^2 - B)(1/X^2 - C) \quad (\text{AI:6})$$

and where A, B, C are the complex roots of the equation

$$N(1/X^2) = 0 \quad (\text{AI:7})$$

Within the range of interest, i.e. $0 \leq X \leq 1$ there is only one real root, say C . Since the coefficients of the equation in (AI:7) are real, the other roots, if complex, are conjugate.

Thus (AI:5) can be rewritten

$$I = \int_{X_1}^{X_2} \frac{\sqrt{1-X^2} (X^2-2k^2)^2}{-4(1-1/k^2)k^4 C(ABX^4-(A+B)X^2+1)(X^2-1/C)} U(|1/X| \pm 1) dx -$$

$$- \int_{X_1}^{X_2} \frac{(1-X^2) \sqrt{k^2-X^2}}{(1-1/k^2)k \cdot C(ABX^4-(A+B)X^2+1)(X^2-1/C)} U(|1/X| - 1/k) dx \quad (AI:8)$$

Singling out the factor $1/(X^2-1/C)$, the remaining part of the integrand becomes a slowly varying function. Denote this part by $H(X)$. Then the integral is approximated by

$$I = H\left(\frac{X_1+X_2}{2}\right) \cdot \int_{X_1}^{X_2} \frac{dx}{X^2-1/C} = H\left(\frac{X_1+X_2}{2}\right) \cdot \frac{\sqrt{C}}{2} \log \left| \frac{(X_2-1/\sqrt{C})(X_1+1/\sqrt{C})}{(X_2+1/\sqrt{C})(X_1-1/\sqrt{C})} \right| \quad (AI:9)$$

In the present calculations the interval $0 \leq X \leq 1$ has been subdivided into 1000 segments and I has been evaluated from (AI:9) for each one. The cumulative sums been calculated as well giving a table of integrals

$$\int_0^{X_A} \frac{h(X)}{X} dx \quad (AI:10)$$

for the argument $0 \leq X_A \leq 1$.

It is then immediately clear that the original problem of evaluating the integral in (AI:1) is solved by making the four appropriate entries as given in (AI:3) in the table for (AI:10). This method of evaluation holds in an overwhelming majority of cases, the only exception being when some integration limit in (AI:3) falls within the very segment that contains $1/\sqrt{C}$. In such a case the last term in the corresponding cumulative sum is calculated from (AI:9).

Appendix II

The displacement u_z due to a uniform pressure q acting on the infinite strip $|x| < vt$ of the surface of a linear elastic semi-infinite solid

BROBERG (1960) has given an analytical expression for the acceleration $\frac{\partial^2 u_z}{\partial \tau^2}$.

Integrating this expression twice with respect to time should give u_z .

$$\frac{\partial^2 u_z}{\partial \tau^2} = - \frac{q}{2\pi k^4 \rho c_d |x|} \left[\frac{\beta}{1 - \beta^2 \frac{\tau^2}{x^2}} \operatorname{Re} \left\{ F \left(\frac{\tau^2}{x^2} \right) \right\} + \frac{\pi}{2} \frac{|x|}{\tau} \beta \operatorname{Im} \left\{ F \left(1/\beta^2 \right) \right\} \cdot \delta \left(\beta - \frac{|x|}{\tau} \right) \right] \quad (\text{AII.1})$$

where $\beta = v/c_d$, that is the ratio between the velocity of the expansion of the strip and the velocity of the irrotational waves,

δ is the Dirac delta-function,

$$\tau = c_d t$$

Other quantities are defined in the main text.

Integrating (AII.1) once gives

$$\frac{\partial u_z}{\partial \tau} = - c \int_{|x|}^{\tau^2} \frac{1}{|x|} \left[\frac{\beta}{1 - \beta^2 \frac{\tau_1^2}{x^2}} \operatorname{Re} \left\{ F \left(\frac{\tau_1^2}{x^2} \right) \right\} + \frac{\pi}{2} \frac{|x|}{\tau_1} \beta \operatorname{Im} \left\{ F \left(1/\beta^2 \right) \right\} \delta \left(\beta - \frac{|x|}{\tau_1} \right) \right] d\tau_1 \quad (\text{AII.2})$$

where

$$c = \frac{q}{2\pi k^4 \rho c_d^2}$$

Since

$$\left(\frac{\tau_1^2}{x^2}\right) = \frac{\sqrt{\frac{\tau_1^2}{x^2} - 1}}{\left(\frac{1}{2k^2} - \frac{\tau_1^2}{x^2}\right)^2 + \frac{\tau_1^2}{x^2} \sqrt{\left(1 - \frac{\tau_1^2}{x^2}\right)\left(1/k^2 - \frac{\tau_1^2}{x^2}\right)}} \quad (\text{A II.3})$$

It is obvious that the second term of the integrand is identically zero as long as $\beta < k$, i.e. the velocity v is smaller than that of the equivoluminal waves.

Initially interest is focussed on the first term of the integrand.

Introducing

$$\frac{\tau_1}{|x|} = T_1$$

$$\text{Re}\{F(T_1^2)\} = \begin{cases} 0 & T_1 < 1 \\ \frac{(T_1^2 - 1/2k^2)^2 \sqrt{T_1^2 - 1}}{(T_1^2 - 1/2k^2)^4 - T_1^4 (T_1^2 - 1)(T_1^2 - 1/k^2)} & 1 \leq T_1 < 1/k \\ \frac{(T_1^2 - 1/2k^2)^2 \sqrt{T_1^2 - 1} + T_1^2 (T_1^2 - 1) \sqrt{T_1^2 - 1/k^2}}{(T_1^2 - 1/2k^2)^4 - T_1^4 (T_1^2 - 1)(T_1^2 - 1/k^2)} & T_1 \geq 1/k \end{cases}$$

The first part of (AII.2) reads

$$\frac{\partial u_z}{\partial \tau} = -c \left[\int_1^{\tau_2/|x|} \frac{\beta}{1-\beta^2 T_1^2} \cdot \frac{\sqrt{T_1^2 - 1} (T_1^2 - 1/2k^2)^2}{N(T_1^2)} U(|T_1| - 1) dT_1 + \right. \\ \left. + \int_1^{\tau_2/|x|} \frac{\beta}{1-\beta^2 T_1^2} \cdot \frac{T_1^2 (T_1^2 - 1) \sqrt{T_1^2 - 1/k^2}}{N(T_1^2)} U(|T_1| - 1/k) dT_1 \right] \quad (\text{AII.4})$$

The denominator in the expression (AII.4)

$$N(T_1^2) = (T_1^2 - 1/2k^2)^4 - T_1^4 (T_1^2 - 1)(T_1^2 - 1/k^2)$$

out equal to zero gives a cubic equation in T_1^2 which written on normal form is

$$(T_1^2)^3 + (T_1^2)^2 \frac{3-2k^2}{2k^2(k^2-1)} + T_1^2 \left(-\frac{1}{2k^4(k^2-1)} \right) + \frac{1}{16k^6(k^2-1)} = 0$$

The criterion for this equation to have three real different roots is

$$\Delta < 0 \quad (\text{AII.5})$$

where

$$\Delta = \left(\frac{d_1}{3} \right)^3 + \left(\frac{d_2}{2} \right)^2$$

and where

$$d_1 = -\frac{a^2}{3} + b$$

and

$$d_2 = 2 \cdot \left(\frac{a}{3} \right)^3 - \frac{ab}{3} + c$$

Using the standard notation

$$x^3 + ax^2 + bx + c = 0$$

Within the range of physical interest, i.e. $0 < k^2 < 0.5$ the condition (AII.5) is

fulfilled for

$$0.32 < k^2 < 0.5$$

The limits of this allowable range correspond to

$$0.26 > \nu > 0$$

$$\text{as } k^2 = \frac{c_r^2}{c_d^2} = \frac{\frac{E}{\rho} \frac{1}{2(1+\nu)}}{\frac{E}{\rho} \frac{1-\nu}{(1+\nu)(1-2\nu)}} = \frac{1-2\nu}{2(1-\nu)} \quad (\text{AII.6})$$

The value chosen in the numerical calculations is $k^2 = 1/3$, that is $\nu = 1/4$, unless otherwise specified.

Taking the step-function into account, modifying the integration limits

(AII.4) is written

$$\frac{\partial u_z}{\partial \tau} = -c \left[\int_1^{\tau_2/|x|} \frac{\beta \sqrt{T_1^2 - 1} (T_1^2 - 1/2k^2)^2}{-\beta^2(T_1^2 - 1/\beta^2)(1 - 1/k^2)(T_1^2 - A)(T_1^2 - B)(T_1^2 - C)} dT_1 + \right. \\ \left. + \int_{1/k}^{\tau_2/|x|} \frac{\beta T_1^2(T_1^2 - 1)\sqrt{T_1^2 - 1/k^2}}{-\beta^2(T_1^2 - 1/\beta^2)(1 - 1/k^2)(T_1^2 - A)(T_1^2 - B)(T_1^2 - C)} dT_1 \right] = -c(I_1 + I_2) \quad (\text{AII.7})$$

The square-roots involved in the numerator of each integrand are eliminated by a substitution of variables

In I_1 the variable t_1 is introduced

$$t_1 = \sqrt{\frac{T_1 - 1}{T_1 + 1}}$$

Since $0 \leq t_1 \leq 1$ in the interval considered the expression giving I_1 becomes

$$I_1 = - \frac{1}{(1 - 1/k^2)\beta k^4(1 - A)(1 - B)(1 - C)(1 - 1/\beta^2)}.$$

$$\int \frac{2t_1^2(1 - t_1^2) \left[(1 + t_1^2)^2 \cdot 2k^2 - (1 - t_1^2)^2 \right]^2 dt_1}{n_-(t_1, A) \cdot n_+(t_1, A) \cdot n_-(t_1, B) \cdot n_+(t_1, B) \cdot n_-(t_1, C) \cdot n_+(t_1, C) \cdot n_-(t_1, \bar{\beta}^2) \cdot n_+(t_1, \bar{\beta}^2)} \quad (\text{AII.8})$$

$$\text{where } n_-(t_1, A) = t_1^2 + \frac{1 - \sqrt{A}}{1 + \sqrt{A}}$$

$$n_+(t_1, A) = t_1^2 + \frac{1 + \sqrt{A}}{1 - \sqrt{A}}$$

and analogously for B, C and $1/\beta^2$.

The numerator being a polynomial in t_1^2 and the denominator being of the form

$\prod_{i=1}^8 (t_1^2 + d_i)$ and having higher degree, a partial fraction expansion can be performed, giving

$$I_1 = \sum_{i=1}^8 \int_0^t \frac{c_i}{t_1^2 + d_i} \quad (\text{AII.9})$$

$$t = \sqrt{\frac{\tau/|x| - 1}{\tau/|x| + 1}}$$

Working completely analogous in I_2 where

$$u_1 = \sqrt{\frac{T_1 - 1/k}{T_1 + 1/k}}$$

is introduced will result in

$$I_2 = - \frac{1}{(1 - 1/k^2)\beta(1 - k^2A)(1 - k^2B)(1 - k^2C)(1 - k^2/\beta^2)} \cdot$$

$$\int_0^u \frac{8k^2 u_1^2 (1+u_1^2)^2 [(1+u_1^2)^2 - k^2(1-u_1^2)^2] (1-u_1^2) du_1}{m_-(u_1, A) \cdot m_+(u_1, A) \cdot m_-(u_1, B) \cdot m_+(u_1, B) \cdot m_-(u_1, C) \cdot m_+(u_1, C) \cdot m_-(u_1, \bar{\beta}^2) \cdot m_+(u_1, \bar{\beta}^2)} \quad (\text{AII.10})$$

$$\text{where } m_-(u_1, A) = u_1^2 + \frac{1-k\sqrt{A}}{1+k\sqrt{A}}$$

$$m_+(u_1, A) = u_1^2 + \frac{1+k\sqrt{A}}{1-k\sqrt{A}}$$

and analogously for B, C and $1/\beta^2$

$$I_2 = \sum_{i=1}^8 \int_0^u \frac{e_i}{u_1^2 + f_i} \quad (\text{AII.11})$$

$$\text{where } u = \sqrt{\frac{\tau/|x| - 1/k}{\tau/|x| + 1/k}}$$

Even if the quantities c_i , d_i , e_i and f_i may be worked out on closed form it is more convenient to fix the k^2 and β -values and then to calculate $c_i \dots f_i$ on a computer. The integrals having the form $\int \frac{a}{x^2+b}$ are easily evaluated. In fact

$$\int \frac{a}{x^2+b} dx = \begin{cases} \frac{a}{\sqrt{b}} \arctan \frac{x}{\sqrt{b}} & b > 0 \\ \frac{a}{2\sqrt{|b|}} \log \left| \frac{x-\sqrt{|b|}}{x+\sqrt{|b|}} \right| & b < 0 \end{cases} \quad (\text{AII.12})$$

Since the lower integration limit is 0 in I_1 and I_2 the result is simple

Thus

$$I_1 = \sum_{d_i^+} \frac{c_i}{\sqrt{d_i}} \arctan \frac{t}{\sqrt{d_i}} + \sum_{d_i^-} \frac{c_i}{2\sqrt{|d_i|}} \log \left| \frac{t-\sqrt{|d_i|}}{t+\sqrt{|d_i|}} \right| \quad (\text{AII.13})$$

where

$$t = \sqrt{\frac{\frac{\tau_2}{|x|} - 1}{\frac{\tau_2}{|x|} + 1}}$$

and

$$I_2 = \sum_{f_i^+} \frac{e_i}{\sqrt{f_i}} \arctan \frac{u}{\sqrt{f_i}} + \sum_{f_i^-} \frac{e_i}{2\sqrt{|f_i|}} \log \left| \frac{u-\sqrt{|f_i|}}{u+\sqrt{|f_i|}} \right| \quad (\text{AII.14})$$

where

$$u = \sqrt{\frac{\frac{\tau_2}{|x|} - 1/k}{\frac{\tau_2}{|x|} + 1/k}}$$

al where the superscript + and - respectively indicate that the sums should extend over the d_i 's and f_i 's being positive and negative respectively.

The integral of the second term of the original (AII.2) is denoted by I_3

$$I_3 = \int_{|x|}^{\tau_2} \frac{\pi}{2} \frac{1}{\tau_1} \beta \operatorname{Im} \left\{ F(1/\beta^2) \right\} \delta \left(\beta - \frac{|x|}{\tau_1} \right) d\tau_1 \quad (\text{AII.15})$$

The new variable $z = \beta - \frac{|x|}{\tau_1}$ is introduced

$$I_3 = \int_{\beta-1}^{\beta - \frac{|x|}{\tau_2}} \frac{\pi}{2} \beta \operatorname{Im} \left\{ F(1/\beta^2) \right\} \frac{1}{\beta-z} \delta(z) dz = \frac{\pi}{2} \operatorname{Im} \left\{ F(1/\beta^2) \right\} U \left(\beta - \frac{|x|}{\tau_2} \right) \quad (\text{AII.16})$$

Now the first integration is completed

$$\frac{\partial u_z}{\partial \tau} = -c \left[I_1 + I_2 + I_3 \right] \quad (\text{AII.17})$$

To get u_z it is necessary to integrate anew

$$u_z = -c \int_{|x|}^{\tau} \left[I_1 + I_2 + I_3 \right] d\tau_2 \quad (\text{AII.18})$$

The integrals to be evaluated take the forms

$$\int_{|x|}^{\tau} C \arctan \frac{t_2}{D} d\tau_2 \quad (\text{AII.19})$$

$$\int_{|x|}^{\tau} \frac{C}{2} \log \left| \frac{t_2 - D}{t_2 + D} \right| d\tau_2 \quad (\text{AII.20})$$

where

$$t_2 = \sqrt{\frac{\frac{\tau_2}{|x|} - 1}{\frac{\tau_2}{|x|} + 1}} \quad (\text{AII.21})$$

and

$$\int_{|x|/k}^{\tau} E \arctan \frac{u_2}{F} d\tau_2 \quad (\text{AII.22})$$

$$\int_{|x|/k}^{\tau} \frac{E}{2} \log \left| \frac{u_2 - F}{u_2 + F} \right| d\tau_2 \quad (\text{AII.23})$$

where $u_2 = \sqrt{\frac{\frac{\tau_2}{|x|} - 1/k}{\frac{\tau_2}{|x|} + 1/k}} \quad (\text{AII.24})$

introducing (AII.21) in (AII.19) and (AII.20) these expressions take the form

$$4|x|C \int_0^t \frac{t_2}{(1-t_2^2)^2} \arctan \frac{t_2}{D} dt_2 \quad (\text{AII.25})$$

and

$$2|x|C \int_0^t \frac{t_2}{(1-t_2^2)^2} \log \left| \frac{t_2 - D}{t_2 + D} \right| dt_2 \quad (\text{AII.26})$$

where

$$t = \sqrt{\frac{\frac{\tau}{|x|} - 1}{\frac{\tau}{|x|} + 1}}$$

analogously for (AII.22) - (AII.24) gives

$$\frac{4|x|E}{k} \int_0^u \frac{u_2}{(1-u_2^2)^2} \arctan \frac{u_2}{F} du_2 \quad \text{(AII.27)}$$

$$\frac{2|x|E}{k} \int_0^u \frac{u_2}{(1-u_2^2)^2} \log \left| \frac{u_2-F}{u_2+F} \right| du_2 \quad \text{(AII.28)}$$

where

$$u = \sqrt{\frac{\frac{\tau}{|x|} - 1/k}{\frac{\tau}{|x|} + 1/k}}$$

Writing

$$I = \int_0^t \frac{x}{(1-x^2)^2} \arctan \frac{x}{a} dx = \frac{1}{2} \int_0^t \arctan \frac{x}{a} d\left(\frac{1}{1-x^2}\right)$$

Integrating by parts and using (AII.12) finally results in

$$I = \frac{1}{2} \left\{ \left(\frac{1}{1-t^2} - \frac{1}{a^2+1} \right) \arctan \frac{t}{a} + \frac{a}{2(a^2+1)} \log \left| \frac{t-1}{t+1} \right| \right\} \quad \text{(AII.29)}$$

Treating in the same way

$$J = \int_0^t \frac{x}{(1-x^2)^2} \log \left| \frac{x-a}{x+a} \right| dx = \frac{1}{2} \int_0^t \log \left| \frac{x-a}{x+a} \right| d\left(\frac{1}{1-x^2}\right)$$

will give the result

$$J = \frac{1}{2} \left\{ \left(\frac{1}{1-t^2} + \frac{1}{a^2-1} \right) \log \left| \frac{t-a}{t+a} \right| + \frac{a}{1-a^2} \log \left| \frac{t-1}{t+1} \right| \right\} \quad \text{(AII.30)}$$

What remains is to perform the integration

$$\int_{|x|}^{\tau} I_3 d\tau_2 = \frac{\pi}{2} \operatorname{Im} \left\{ F(1/\beta^2) \right\} \int_{|x|}^{\tau} U(\beta - \frac{|x|}{\tau_2}) d\tau_2 \quad (\text{AII.31})$$

Again introducing

$$z = \beta - \frac{|x|}{\tau_2}$$

(AII.31) reads

$$\begin{aligned} \frac{\pi}{2} \operatorname{Im} \left\{ F(1/\beta^2) \right\} \int_{\beta-1}^{\beta - \frac{|x|}{\tau}} U(z) \cdot \frac{|x|}{(\beta-z)^2} dz &= \frac{\pi}{2} \operatorname{Im} \left\{ F(1/\beta^2) \right\} |x| \int_0^{\beta - \frac{|x|}{\tau}} \frac{1}{\beta-z} dz \\ &= \frac{\pi}{2} \operatorname{Im} \left\{ F(1/\beta^2) \right\} \frac{|x|}{\beta} \left(\frac{\beta}{\frac{|x|}{\tau}} - 1 \right) \end{aligned} \quad (\text{AII.32})$$

Now u_z is completely determined

$$u_z = -c \int_{|x|}^{\tau} (I_1 + I_2 + I_3) d\tau_2$$

Dividing both sides by a reference length a the non-dimensional displacement

$v_z = u_z/a$ finally reads:

$$\begin{aligned}
 v_z = & -c |\xi| \left\{ 2 \sum_{d_i^+} \frac{c_i}{\sqrt{d_i}} \left[\left(\frac{1}{1-t^2} - \frac{1}{d_i+1} \right) \arctan \frac{t}{\sqrt{d_i}} + \frac{\sqrt{d_i}}{2(d_i+1)} \log \left| \frac{t-1}{t+1} \right| \right] + \right. \\
 & + \sum_{d_i^-} \frac{c_i}{\sqrt{|d_i|}} \left[\left(\frac{1}{1-t^2} + \frac{1}{|d_i|-1} \right) \log \left| \frac{t-\sqrt{|d_i|}}{t+\sqrt{|d_i|}} \right| + \frac{\sqrt{|d_i|}}{1-|d_i|} \log \left| \frac{t-1}{t+1} \right| \right] + \\
 & + \frac{2}{k} \sum_{f_i^+} \frac{e_i}{\sqrt{f_i}} \left[\left(\frac{1}{1-u^2} - \frac{1}{f_i+1} \right) \arctan \frac{u}{\sqrt{f_i}} + \frac{\sqrt{f_i}}{2(f_i+1)} \log \left| \frac{u-1}{u+1} \right| \right] + \\
 & + \frac{1}{k} \sum_{f_i^-} \frac{e_i}{\sqrt{|f_i|}} \left[\left(\frac{1}{1-u^2} + \frac{1}{|f_i|-1} \right) \log \left| \frac{u-\sqrt{|f_i|}}{u+\sqrt{|f_i|}} \right| + \frac{\sqrt{|f_i|}}{1-|f_i|} \log \left| \frac{u-1}{u+1} \right| \right] + \\
 & + \frac{\pi}{2\beta} \operatorname{Im} \left\{ F(1/\beta^2) \right\} \left(\frac{\beta}{|x|/\tau} - 1 \right) \Big\} \quad (\text{AII.33})
 \end{aligned}$$

where

$$\xi = x/a$$

$$t = \sqrt{\frac{\tau/|x| - 1}{\tau/|x| + 1}}$$

$$u = \sqrt{\frac{\tau/|x| - 1/k}{\tau/|x| + 1/k}}$$

and

$$c = \frac{q}{2\pi k^4 \rho c_d^2}$$

Using (AII.6) the constant reads

$$c = \frac{q}{\pi E} \cdot \frac{2(1-v^2)}{1-2v} = \frac{q}{\pi E} \cdot \frac{3-4k^2}{2k^2(1-k^2)}$$

Appendix III

Some aspects on the solution of a Fredholm integral equation of the first kind.

It seems to be generally agreed upon that the solution of the equation

$$\int_a^b K(x,y)f(y)dy = g(x) \quad (\text{AIII:1})$$

usually is a difficult matter, and it is easily seen that for a given kernel K the existence of a solution is not granted for every function g .

To illustrate this, consider a degenerate kernel $K(x,y)=X(x)Y(y)$.

Then (AIII:1) reads

$$\int X(x)Y(y)f(y)dy = g(x)$$

and no solution exists unless g is a multiple of X .

FOX (1962) discusses the case when K is of a general Pincherle-Goursat type, i.e. $K(x,y) = \sum_{r=1}^n X_r(x)Y_r(y)$ which leads to the same kind of restrictions for the existence of a solution.

TRICOMI (1970) has treated the solution of an equation where the kernel primarily is symmetric and secondly belongs to the L_2 class. The solution is given in terms of a eigenvalue-eigenfunction expansion and it is demonstrated that if the series converges, it converges to the solution of the equation.

When it comes to a numerical solution of equations of type (AIII:1) the difficulties are even greater as the problem in the classical sense is incorrectly posed, which not only means that a solution cannot be supposed to exist whatever the right hand side, but also that if a solution is found, it does not necessarily have a continuous dependence on the right hand side. Suppose that the equation $\int Kf = g_1$ is solved and the solution is f_1 . Suppose in addition to this that the equation $\int Kf = g_2$ is solved giving the solution f_2 . If now g_1 and g_2 do not differ much, for example in the sense that $\|g_1 - g_2\| < \epsilon$ it is generally difficult to give a statement about the smallness of $\|f_1 - f_2\|$. Putting it another way, the problem is inherently unstable and ill-conditioned.

Under certain circumstances however it seems to be possible to handle the problem after the application of some kind of a stabilizing process. HAMSTAD and BOGY (1972) in looking for an elastostatic solution for a circular membrane have treated an arising integral equation of the form (AIII:1) by a method described by BAKUSHINSKII (1965).

If (AIII:1) is written in operator form, it reads

$$Af = g \quad (\text{AIII:2})$$

Here it is assumed that A is a completely continuous linear operator. If A in addition to this is positive then it has a complete system of eigenfunctions which correspond to real, non-negative eigenvalues. It can be shown that, provided certain convergence conditions in the eigenfunction expansion are fulfilled

$$\lim_{\alpha \rightarrow 0} \|f_\alpha - f\| = 0 \quad (\text{AIII:3})$$

where f_α is the solution of the equation

$$\alpha f_\alpha + A f_\alpha = g \quad (\text{AIII:4})$$

and where α is any positive number.

In dealing with equations having kernels of a more general type it hardly seems to be the right way to look for direct solution methods giving a high degree of accuracy, particularly if the function g is not known in exact analytical form. This applies especially when g is based on experimental observations or when g due to the method of calculation exhibits an inherent degree of uncertainty.

Thus efforts should be directed towards methods for which, at least as long as problems resting on a physical reality are concerned, a reasonably smooth function f could be found, such that some residual function

$$\epsilon(x) = \int K(x,y)f(y)dy - g(x) \quad (\text{AIII:5})$$

becomes small in some sense throughout the range of x .

BAKUSHINSKII (1965) deals with this problem as well in minimizing the functional $||Af-g||$. The analysis rests on the formation of a positive operator A^*A and handling the equation

$$A^*Af = A^*g \quad (\text{AIII:6})$$

analogously with what was described above. Here $A^*h = \int \bar{K}(y,x)h(y)dy$ where the bar indicates complex conjugation. Thinking about the operators in matrix representation, A^* means the complex conjugate, transpose of A .

HANSON (1971) has suggested an interesting approximative numerical method for solving Fredholm integral equations of the first kind. The method rests on the use of some simple quadrature formula to obtain a matrix equation in which each row is weighted by the reciprocal of the standard deviation of the known function. The resulting linear system is then solved by singular value decomposition. By discarding a number of the smallest singular values it is demonstrated that a solution is obtained which solves the problem "close enough" without increasing the frequently occurring oscillations in the unknown function.

The singular value decomposition is performed in the way described by GOLUB and REINSCH (1969). The computer routines given by GOLUB and REINSCH are written in ALGOL. SÖDERSTRÖM (1970) has translated them into FORTRAN and made them available at the computer center of Lund University.

The real matrix $A_{m \times n}$ is decomposed in the following way

$$A_{m \times n} = U_{m \times n} S_{n \times n} V_{n \times n}^T \quad (\text{AIII:7})$$

where $U^T U = I$, $V V^T = I$ and $S = \text{diag}(s_1, s_2, \dots, s_n)$. Superscript T denotes the transpose of the matrix.

U is made up of n orthonormalized eigenvectors associated with the n largest eigenvalues of $A A^T$ and V consists of the orthonormalized eigenvectors of $A^T A$. The diagonal elements in S are the non-negative square roots of the eigenvalues of $A^T A$ which are called singular values. Supposing the s_i ordered in descending order then $s_{r+1} = s_{r+2} = \dots = s_n = 0$ if $\text{rank}(A) = r$. The decomposition of A in the fashion described is called singular value decomposition.

It should be pointed out that using ordinary diagonalization routines on the symmetric matrix $A^T A$ will involve unnecessary inaccuracy since $A^T A$ has to be formed explicitly. Thus a straightforward application of the BAKUSHINSKII method described above is not likely to be the best way from a computational point of view. Instead interest should be directed towards methods using singular value decomposition.

In singular value decomposition of A the GOLUB and REINSCH routines first use Householder transformations to bring A on bidiagonal form whereupon a QR algorithm finds the eigenvalues of the bidiagonal matrix.

Having rewritten the integral equation on matrix form, say

$$Ax = b \quad (\text{AIII:8})$$

where $m \geq n$ for the matrix $A_{m \times n}$ and where b is the "known" vector, then the vector x , such that

$$\|Ax - b\|_2 = \min \quad (\text{AIII:9})$$

is sought.

If $\text{rank}(A)$ is less than n no unique solution to (AIII:9) exists unless it is required that the x satisfying (AIII:9) also fulfils

$$\|\hat{x}\|_2 = \min \quad (\text{AIII:10})$$

Then $\hat{x}$ is unique. It can be demonstrated, cf. for example BOUILLON and ODELL (1971), that

$$\hat{x} = A^+ b \quad (\text{AIII:11})$$

where A^+ is the pseudoinverse of A .

If A is a real matrix $m \times n$, then a matrix X $n \times m$ is called the pseudoinverse of A if the following four conditions are fulfilled

$$AXA = A$$

$$XAX = X$$

$$(AX)^T = AX$$

$$(XA)^T = XA$$

(AIII:12)

This unique matrix is denoted by A^+ . Now the singular value decomposition of A is handy in the calculation of A^+ . Thus for

$$A = USV^T$$

it can be shown that

$$A^+ = VS^+U^T$$

(AIII:13)

where $S^+ = \text{diag}(s_i^+)$ and

$$s_i^+ = \begin{cases} 1/s_i & s_i > 0 \\ 0 & s_i = 0 \end{cases}$$

Returning to (AIII:11) the "solution" $\hat{x}$ is

$$\hat{x} = A^+b = VS^+U^Tb$$

(AIII:14)

Modifying (AIII:8) according to HANSON's outlines the matrix equation reads

$$PAx = Pb$$

(AIII:15)

where $P = \text{diag}(p_1, p_2, \dots, p_m)$ and where p_i is the factor by which the

i :th equation is weighted. HANSON (1971) suggests the choice

$$p_i = 1/\sigma_i$$

where σ_i is the standard deviation of the i :th component of b .

Referring to (AIII:14) the solution $\hat{x}$ of (AIII:15) in the meaning of (AIII:9) and (AIII:10) is

$$\hat{x} = (PA)^+ Pb \tag{AIII:16}$$

Appendix IV

Numerical evaluation of the stress distribution on the plane of symmetry along a central crack extending at constant relative velocity β_0

BROBERG (1960) has given the expression

$$\sigma(x) = - \int_{\frac{|x|}{\tau}}^1 f(\beta) d\beta \quad (\text{AIV:1})$$

valid for $|x|/\tau > \beta_0$

Here

$$f(\beta) = q_{00} \frac{\beta_0^2 (1 - \beta_0^2)}{g(\beta_0)} \left\{ \frac{(\beta^2 - 2k^2)^2 - 4k^3 \sqrt{(1 - \beta^2) \cdot (k^2 - \beta^2)}}{\beta^2 \sqrt{(1 - \beta^2) \cdot (\beta^2 - \beta_0^2)^3}} + \right. \\ \left. + \frac{4k^3 \sqrt{(1 - \beta^2) \cdot (k^2 - \beta^2)}}{\beta^2 \sqrt{(1 - \beta^2) \cdot (\beta^2 - \beta_0^2)^3}} U(\beta^2 - k^2) \right\} \quad (\text{AIV:2})$$

where U is the unit step function, and

$$g(\beta_0) = ((1 - 4k^2)\beta_0^2 + 4k^4) K(\sqrt{1 - \beta_0^2}) - \\ - \frac{1}{\beta_0^2} (\beta_0^4 - 4k^2(1 + k^2)\beta_0^2 + 8k^4) E(\sqrt{1 - \beta_0^2}) - \\ - 4k^2(1 - \beta_0^2) K(\sqrt{1 - (\beta_0/k)^2}) + \frac{8k^4}{\beta_0^2} (1 - \beta_0^2) E(\sqrt{1 - (\beta_0/k)^2}) \quad (\text{AIV:3})$$

In the expression for g , K and E denote the complete elliptic integrals of the first and second kind respectively.

The K :s and E :s are calculated by means of an accurate fourth-order polynomial approximation given by ABRAMOWITZ and STEGUN (1970). The integral in (AIV:1) is then evaluated by a straightforward trapezoidal rule. A picture of the results obtained is found in Fig. 6.

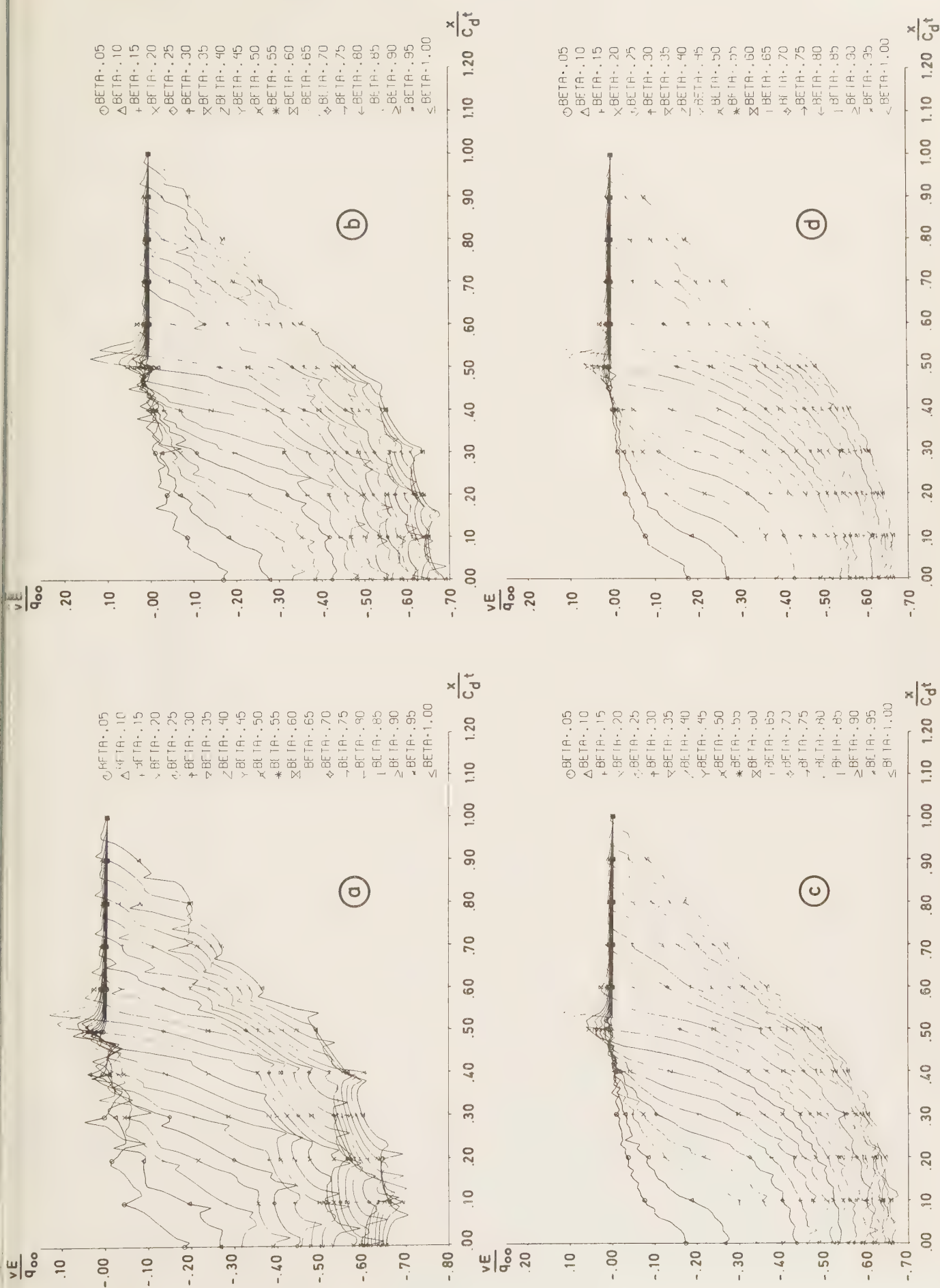


Fig.1. Non-dimensional displacement vE/q_{∞} due to a box expanding at constant relative velocity β . $\beta=0.05(0.05)1.0$, $k^2=1/3$, i.e. $v=0.25$.

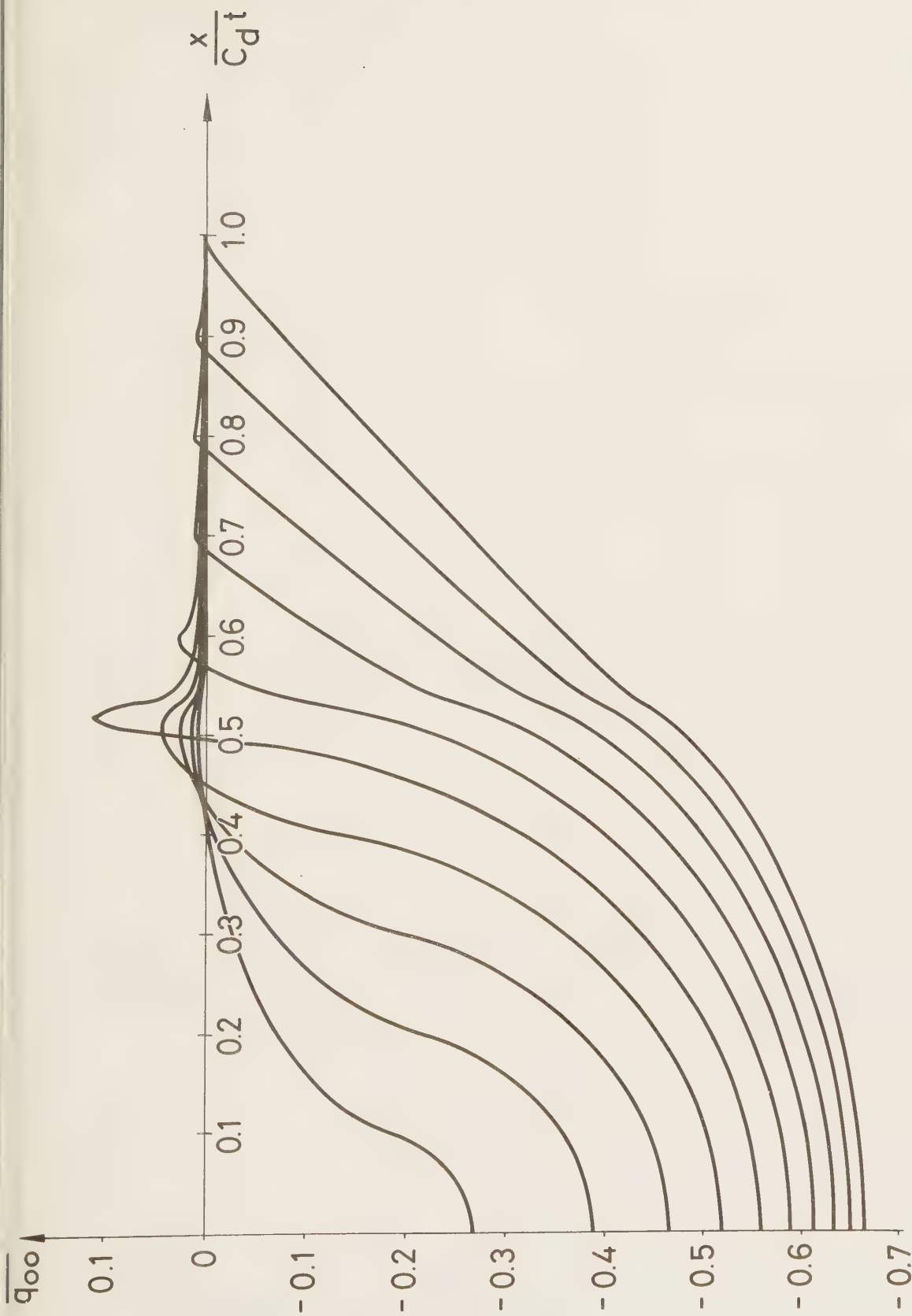


Fig.2. Non-dimensional displacement vE/q_{∞} due to a box expanding at constant relative velocity β .

$\beta=0.1(0.1)1.0$, (the lowest curve corresponds to $\beta=1.0$), $k^2=1/3$, i.e. $v=0.25$

These curves for $n=200$ cannot be resolved from the exact solution in Appendix II on the scale of the figure.

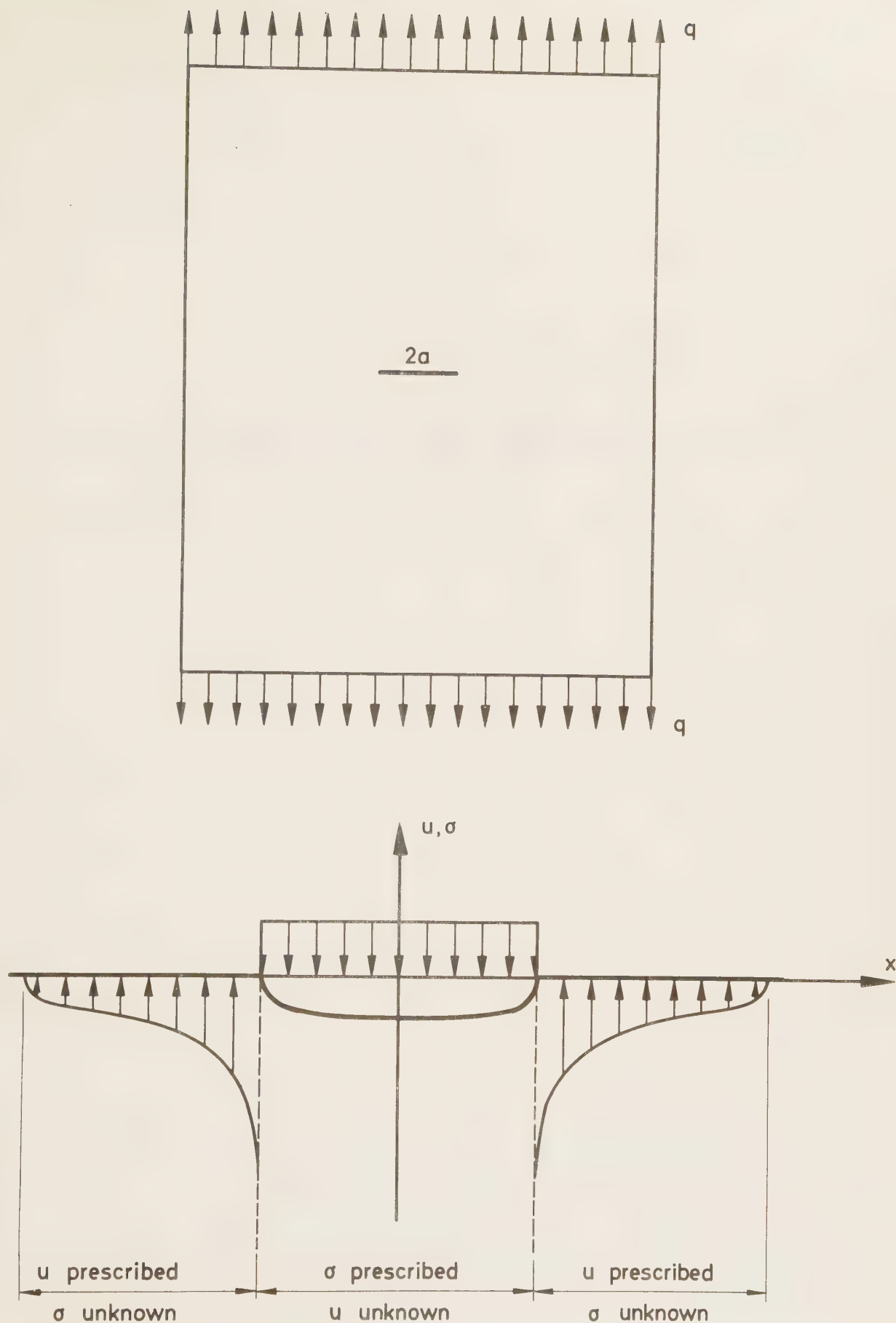


Fig.3. The mixed boundary value problem equivalent to that of a central crack loaded in the extensional mode.

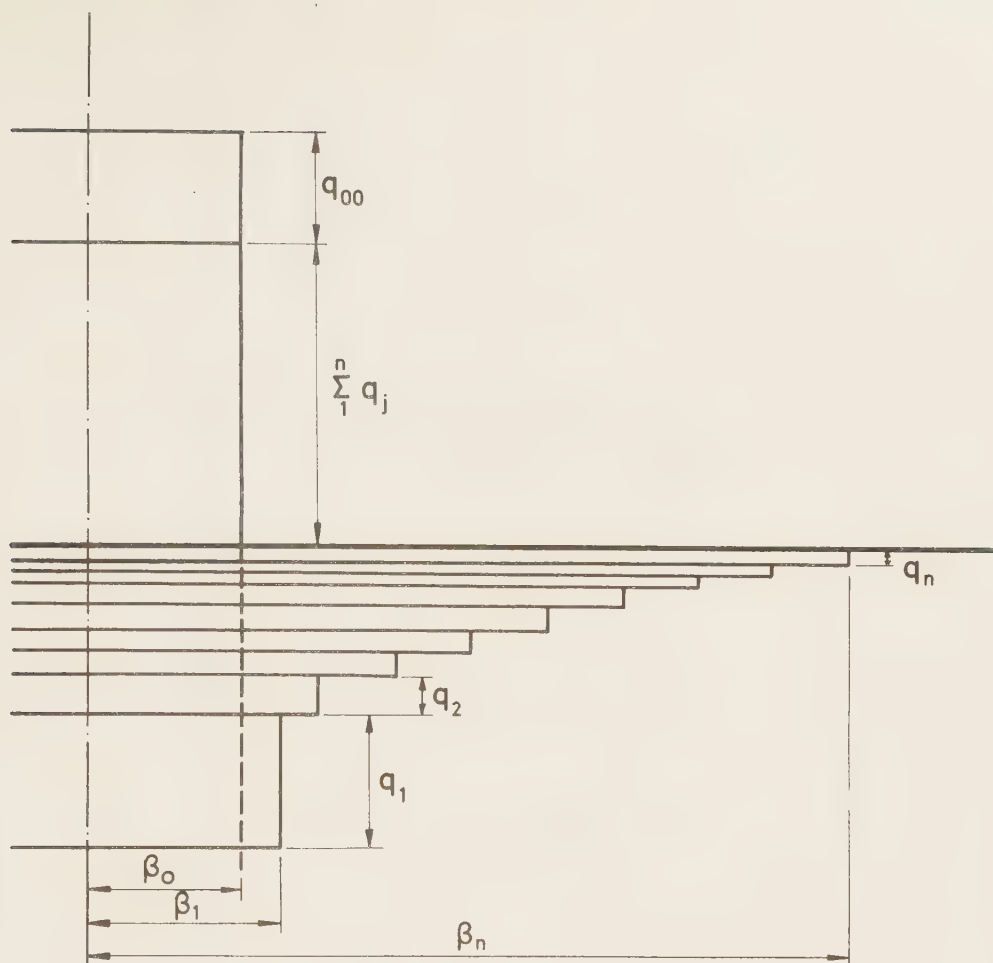


Fig.4. The box model in the constant velocity case.

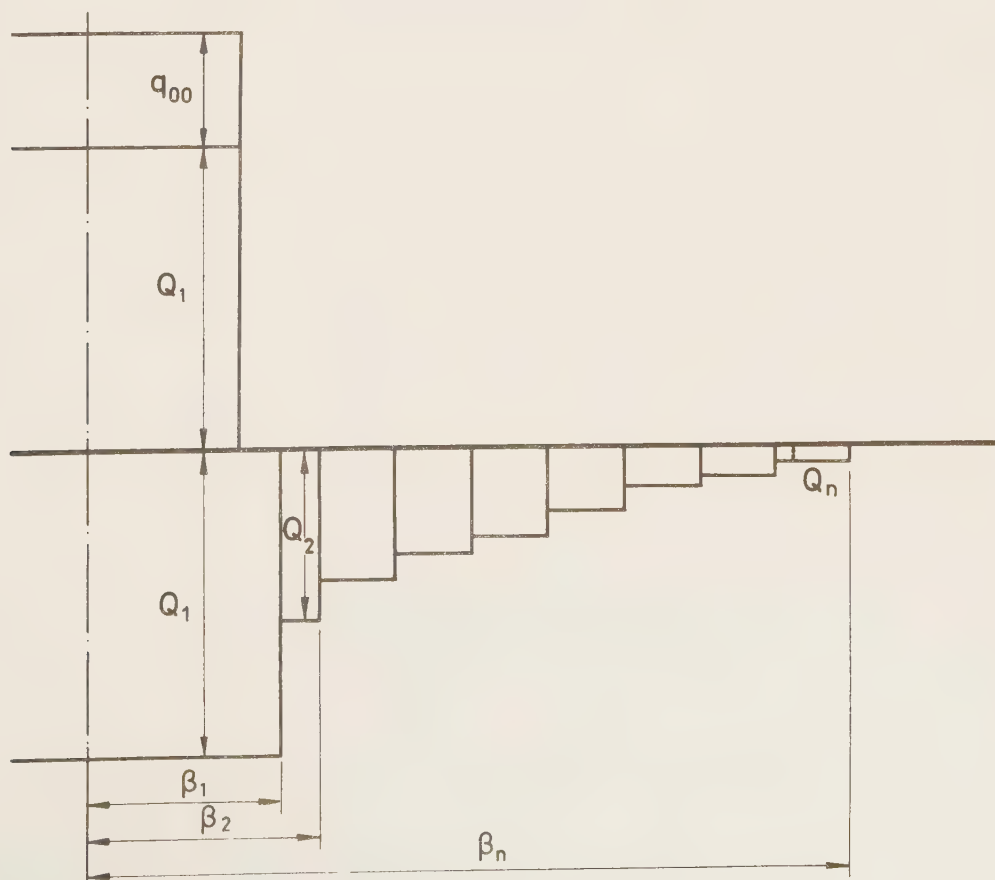


Fig.5. The stress distribution related to the amplitude of the expanding boxes.

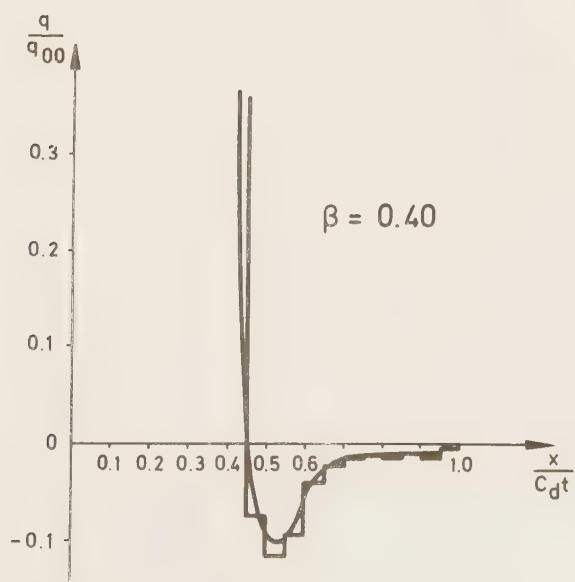
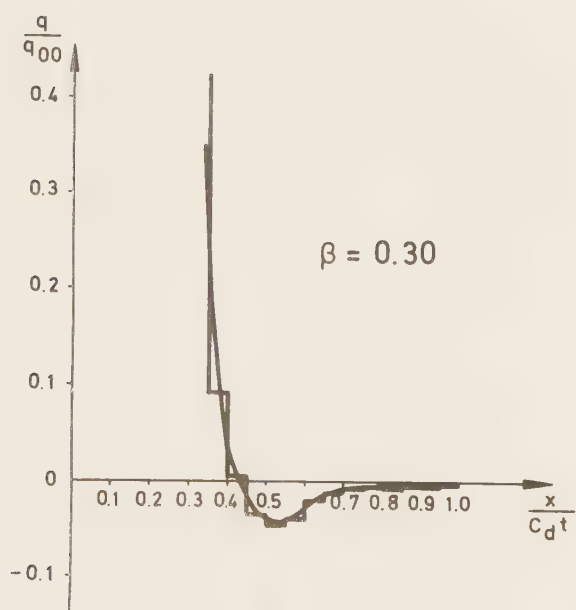
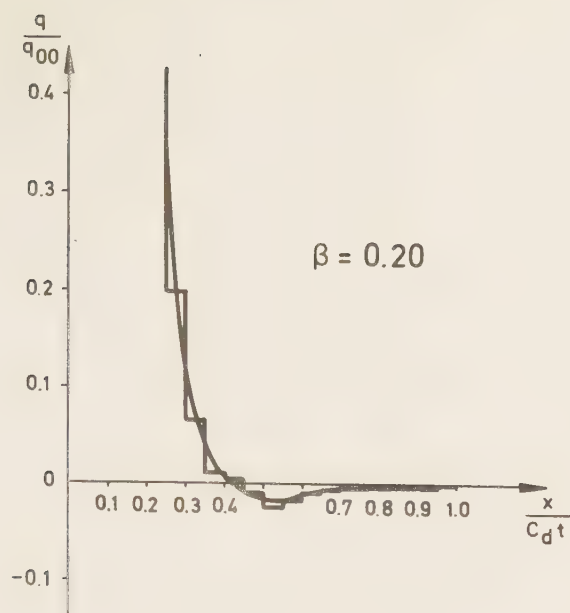


Fig.6. Stress distribution for a crack extending at constant relative velocity β_0
a: $\beta_0 = 0.20$ b: $\beta_0 = 0.30$ c: $\beta_0 = 0.40$. $k^2 = 1/3$, i.e. $v = 0.25$
Smooth curves: The exact solution from BROBERG (1960), evaluated in Appendix IV.

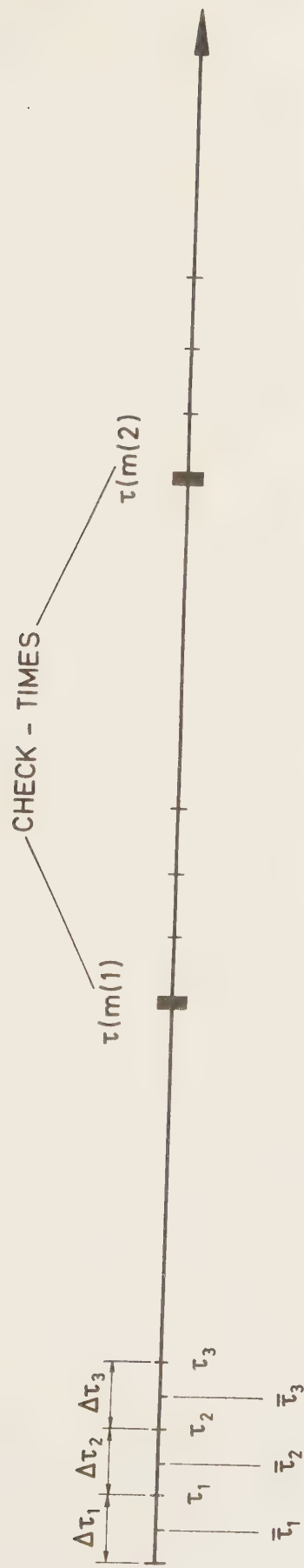


Fig.7. Check times $\tau(m(i))$ related to time increments $\Delta\tau$.

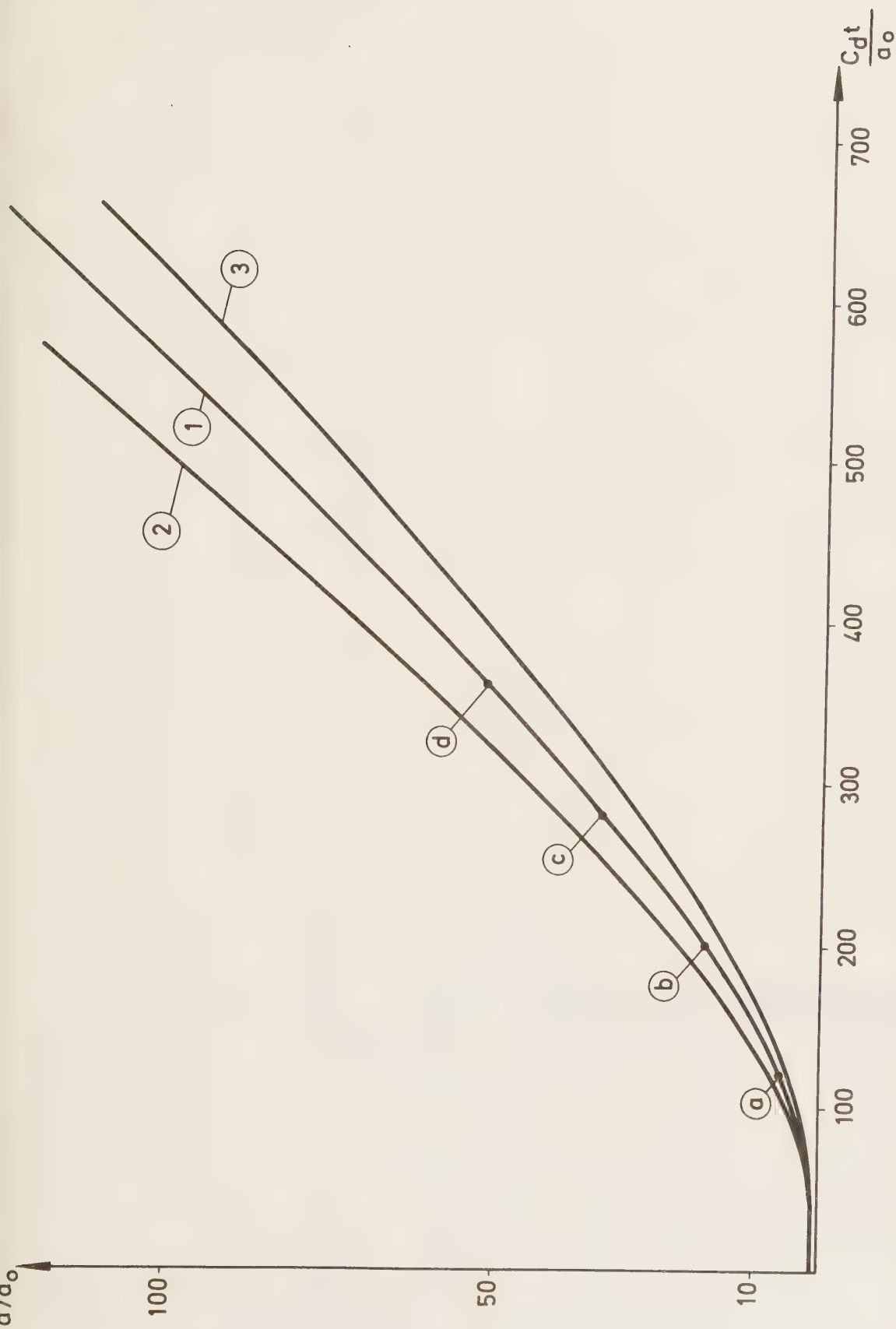


Fig.8. Non-dimensional crack length a/a_0 vs. non-dimensional time $C_d t / a_0$. 1: From BERGKVIST (1973). The stress distributions at a, b, c and d respectively are found in Fig.9.

2: Faster moving crack; a given length reached in 0.9 of the time required in case 1.

3: slower crack; a given length reached in 1.1 of the time required in case 1.

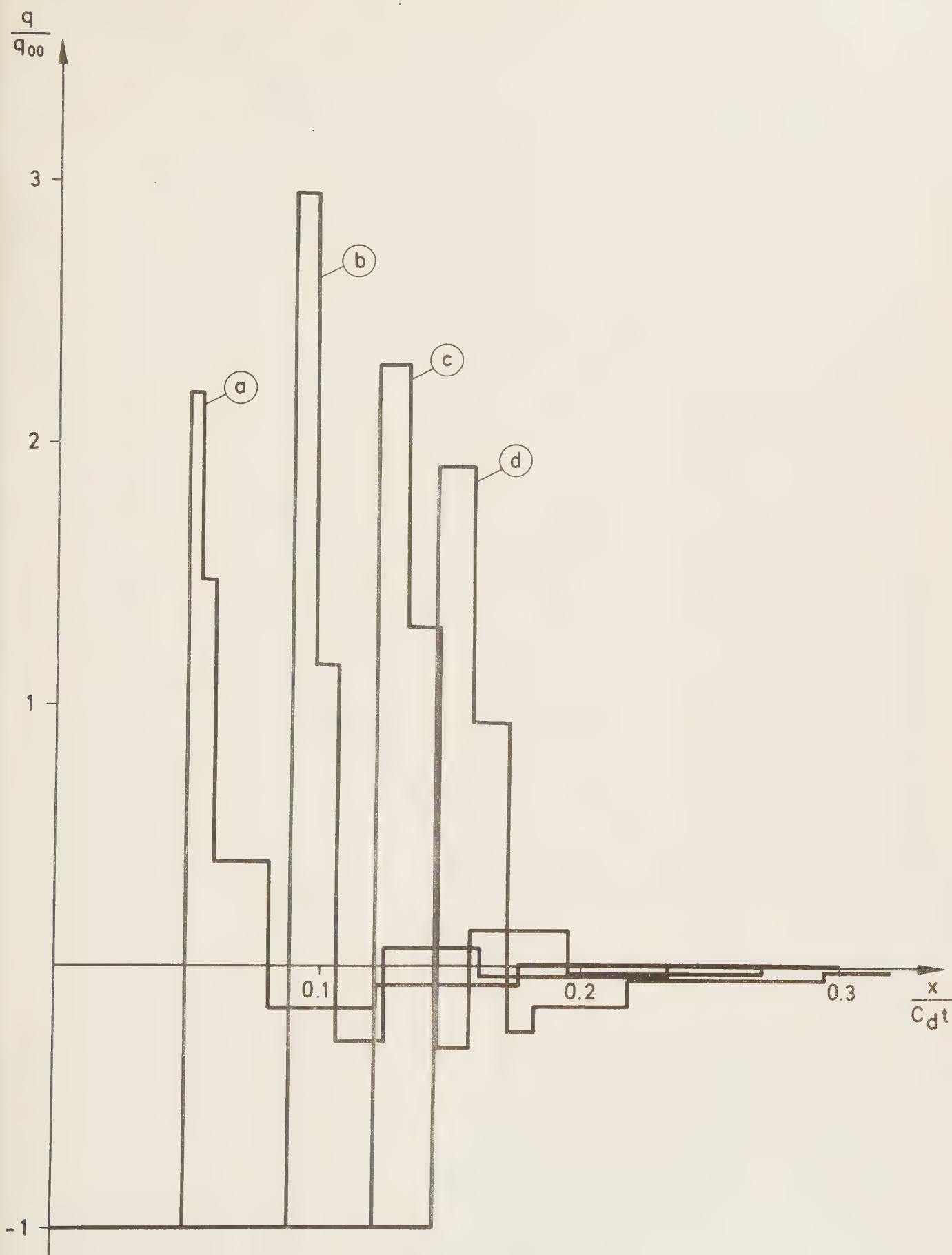


Fig.9. Stress distribution near the crack-tip at different stages of crack extension. $k^2=1/6$, i.e. $\nu=0.4$

$$a: \frac{c_d t}{a_0} = 120, \quad b: \frac{c_d t}{a_0} = 200, \quad c: \frac{c_d t}{a_0} = 280, \quad d: \frac{c_d t}{a_0} = 360$$

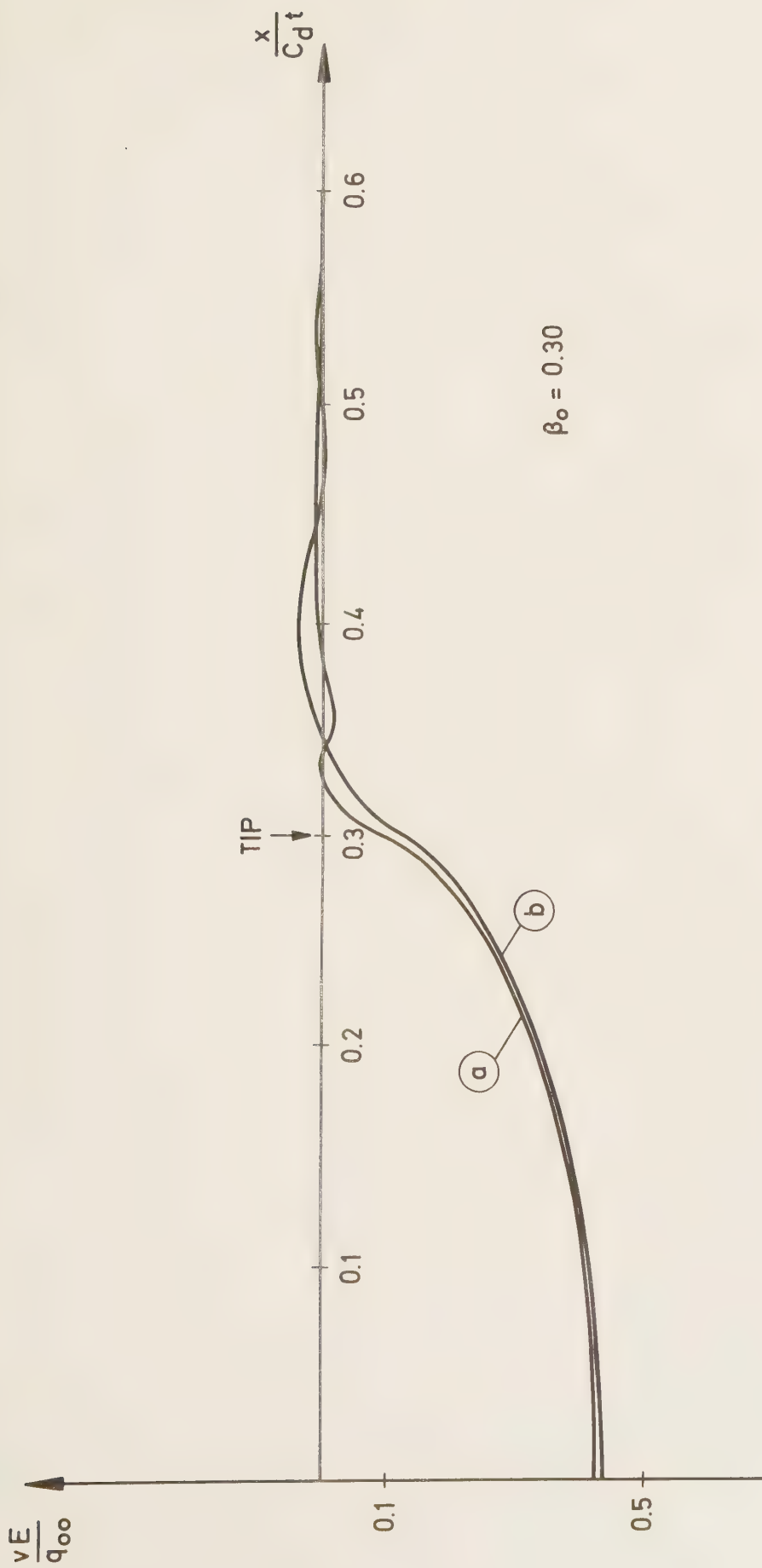


Fig. 10. The crack shape for a crack extending at constant relative velocity $\beta_0 = 0.30$. $k^2 = 1/3$, i.e. $v = 0.25$
a: all singular values retained, b: the smallest singular value discarded.

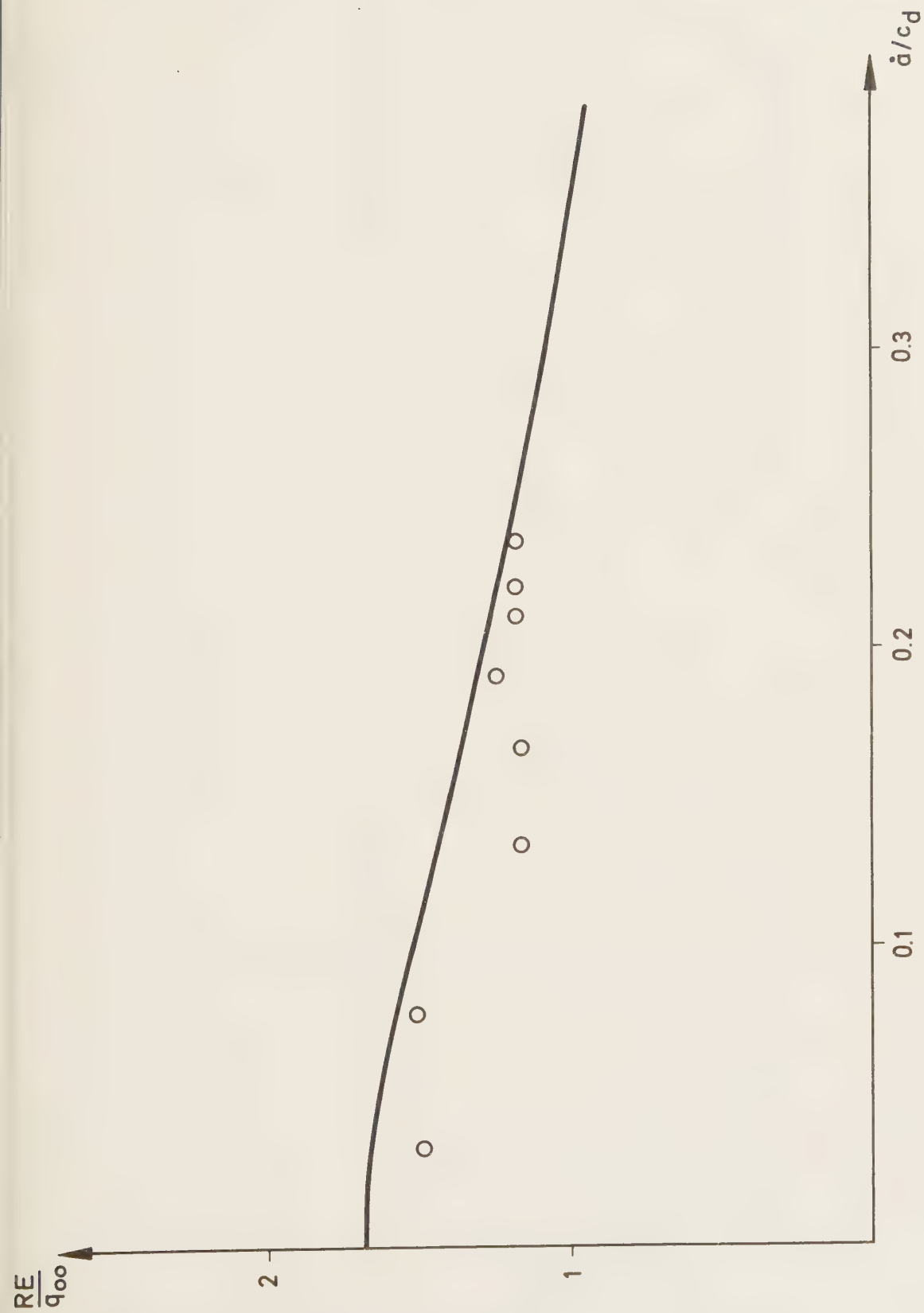


Fig. 11. The ratio R between the shorter and the longer diagonals of the crack plotted vs. non-dimensional crack-tip velocity (dots). $k^2 \approx 1/6$, i.e. $v \approx 0.4$
 Solid curve for the constant velocity case, eq (40).

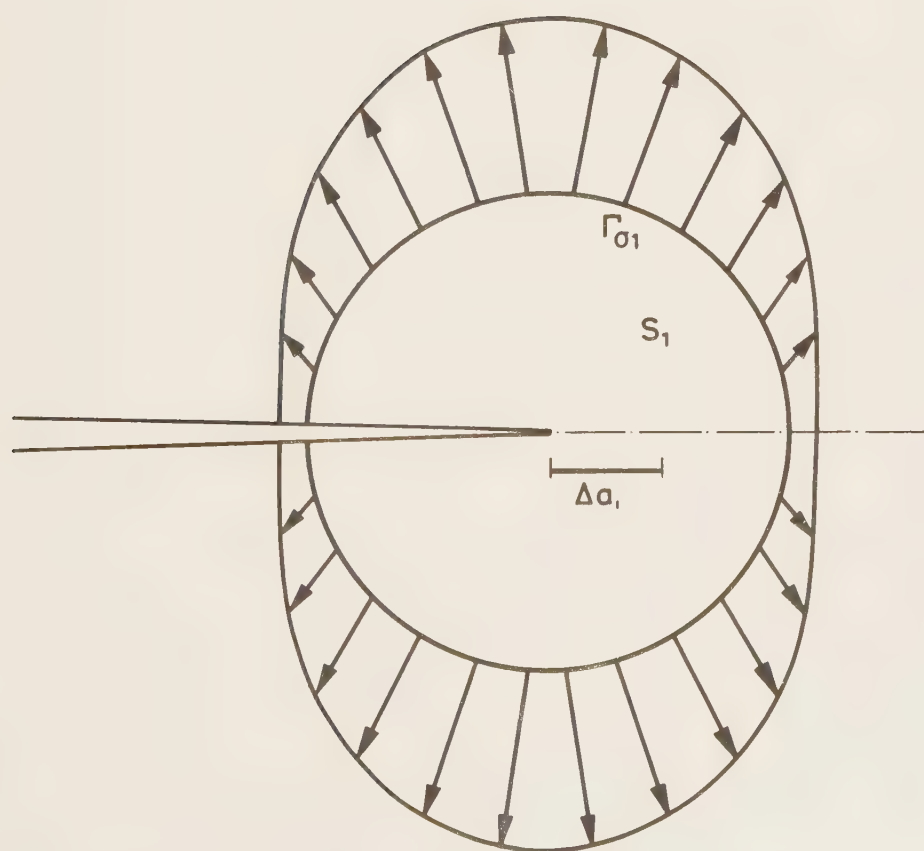
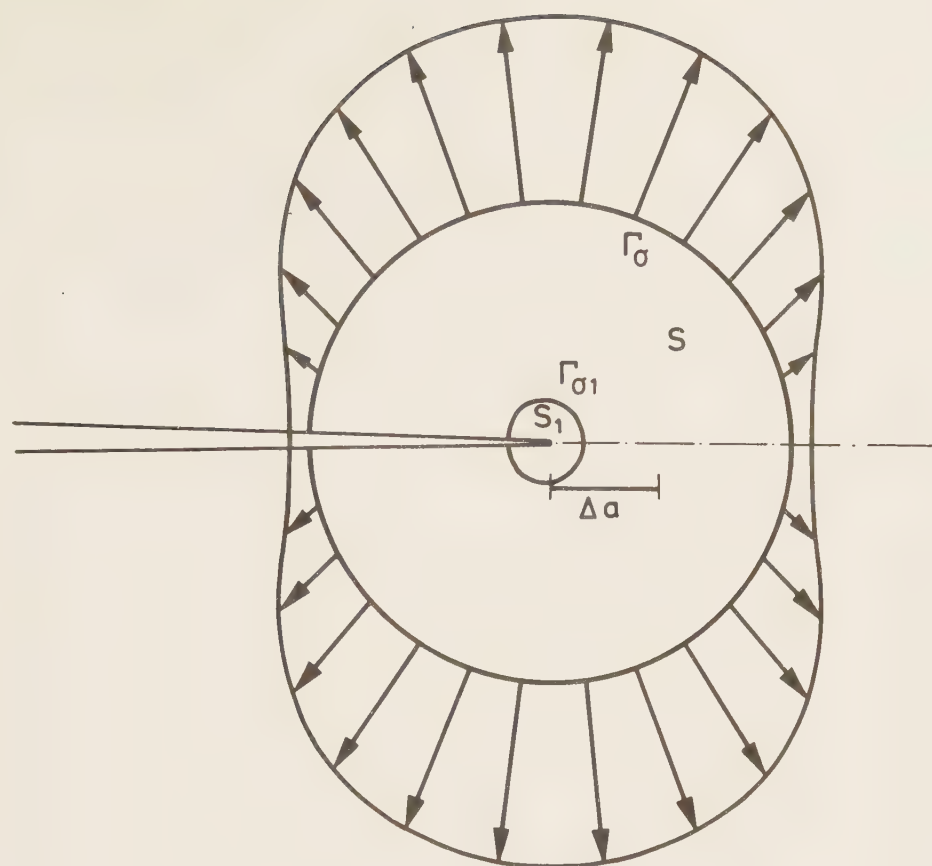


Fig.12. Situation in the neighbourhood of the crack-tip.

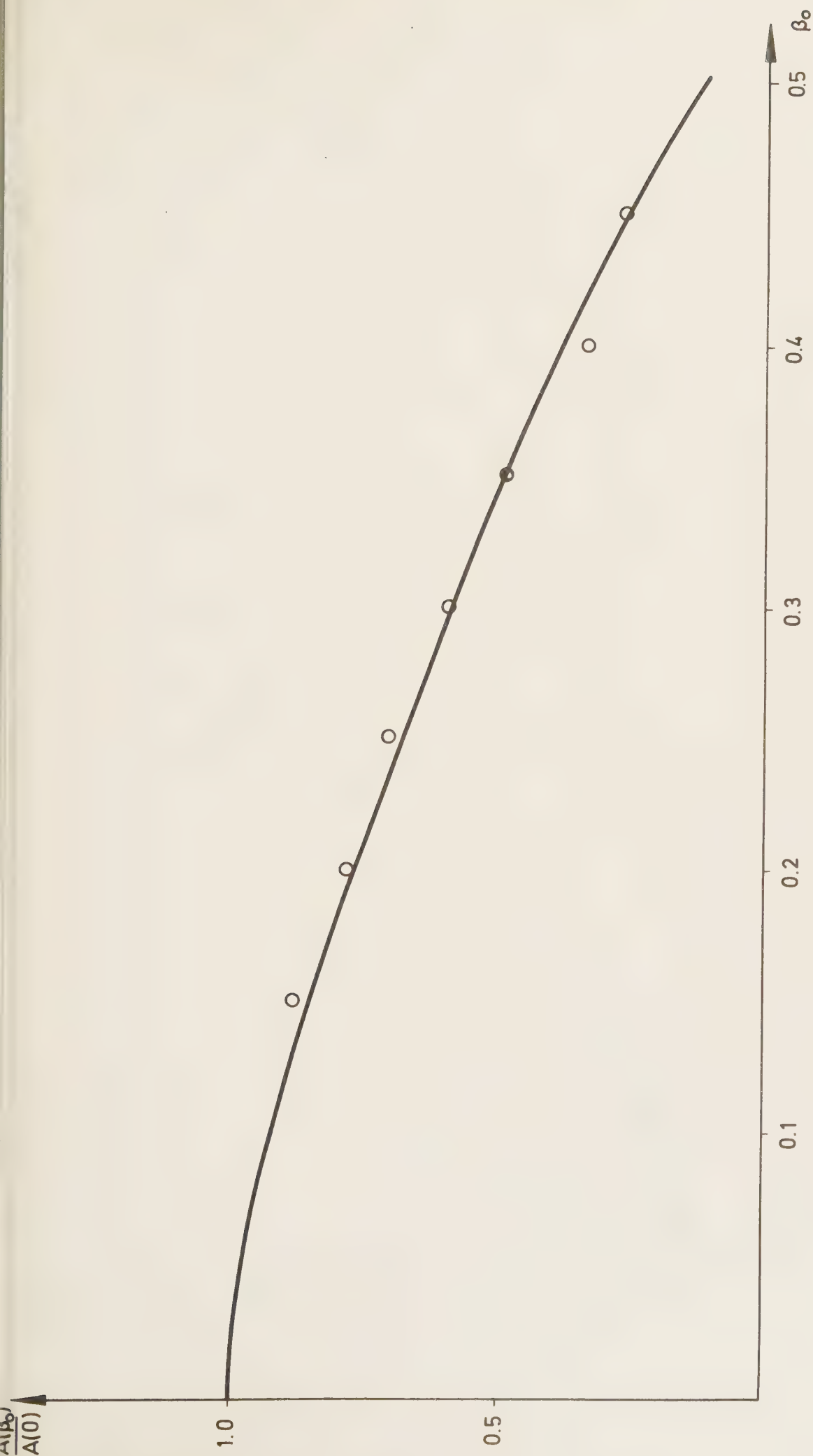


Fig. 13. Non-dimensional stress intensity factor vs. crack-tip velocity in the constant velocity case (dots).
 $k^2=1/3$, i.e. $\nu=0.25$. Solid curve, eq (45)

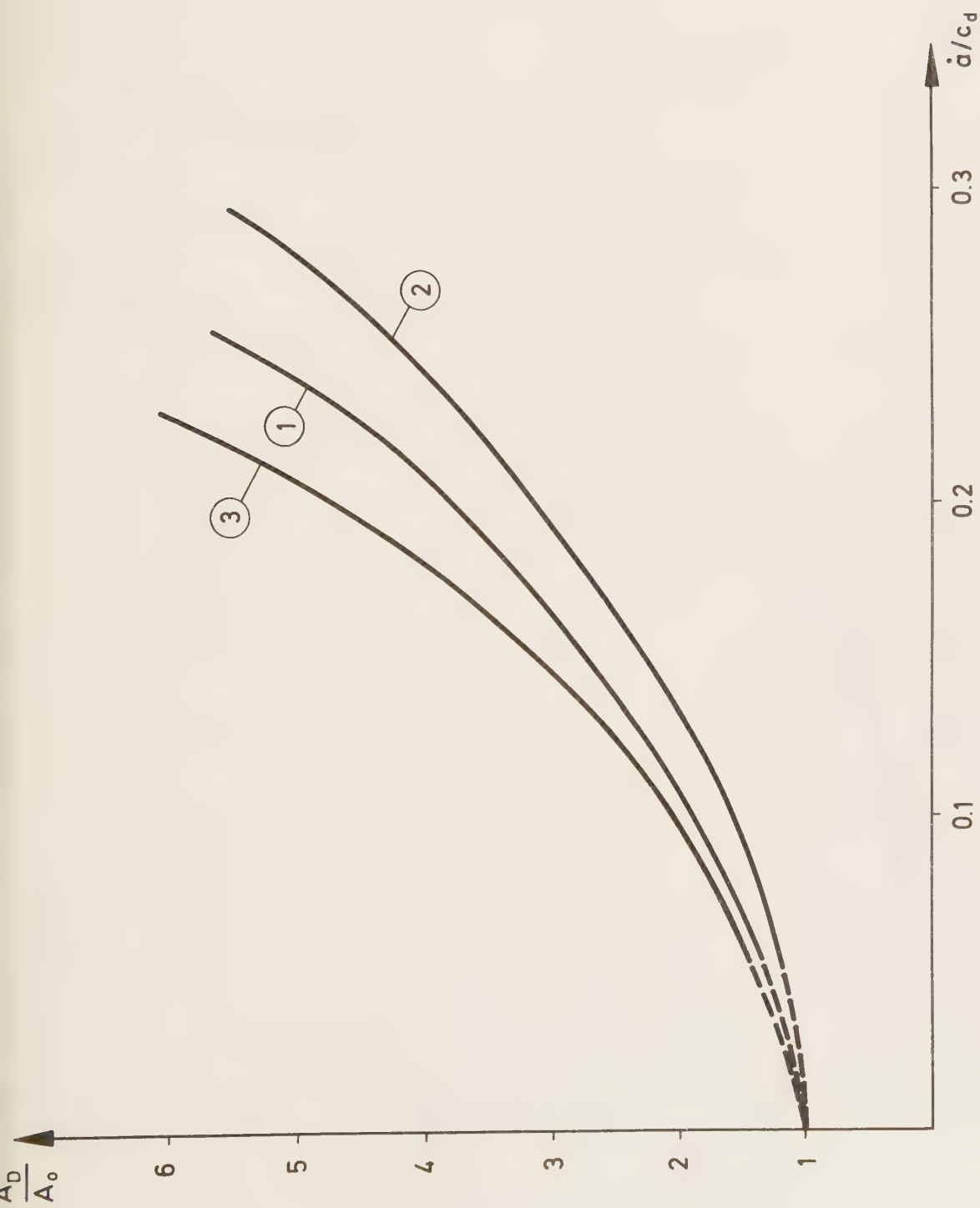


Fig. 14. Dynamic stress intensity factor A_D normalized with respect to the static stress intensity factor A_0 for the original crack length vs. instantaneous crack-tip velocity $\dot{a}/c_d$. The crack-tip motion in case 1, 2 and 3 is given in Fig. 8.

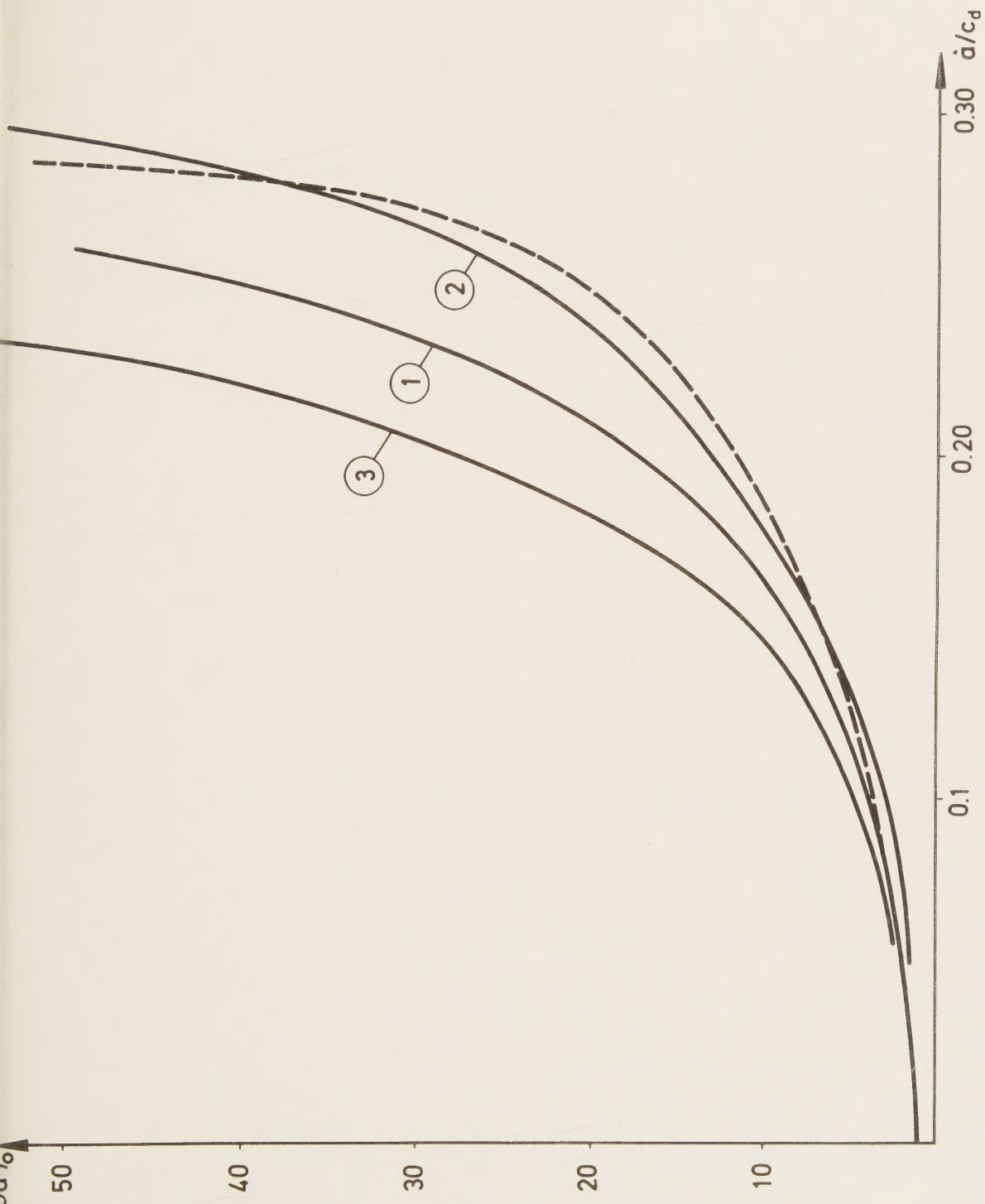


Fig.15. The release of energy per unit of crack extension, normalized with respect to the energy release at the original static state, plotted vs. instantaneous crack-tip velocity $\dot{a}/c_d$. Crack motion for case 1, 2 and 3 is given in Fig.8.

The broken line gives normalized fracture energy $\Gamma = \Gamma(\dot{a}/c_d)$ (from PAXSON and LUCAS (1972))

